Dependency Graph Method for Proving Termination of Narrowing ⋆

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Abstract. Term rewriting systems with extra variables are useful in encoding operators for inverse computation. Their ground rewrite sequences can be simulated by narrowing sequences. In this paper, we refine the dependency pair method for proving termination of narrowing and extend the dependency graph method for proving termination of rewriting to a method for narrowing.

1 Introduction

Term rewriting systems with extra variables (*EV-TRSs*, for short), i.e., sets of rewrite rules that may have extra variables, are useful in encoding operators that compute inverse images of the corresponding functions [12,13]. Here, *extra variables* of a rewrite rule are variables appearing not in the left-hand side but in the right-hand side. For example, the following EV-TRS is a part of the system automatically generated by inversion compilers [12,13] from a program for computing multiplication of natural numbers:

$$R_1 = \left\{ \begin{array}{ll} \overline{\mathsf{mul}}(\mathsf{0}) \to \mathsf{tp}_2(\mathsf{0}, y), & \overline{\mathsf{mul}}(\mathsf{0}) \to \mathsf{tp}_2(x, \mathsf{0}), \\ \overline{\mathsf{mul}}(\mathsf{s}(z)) \to \mathsf{u}_2(\overline{\mathsf{add}}(z)), & \mathsf{u}_2(\mathsf{tp}_2(w, y)) \to \mathsf{u}_3(\overline{\mathsf{mul}}(w), y), \\ \mathsf{u}_3(\mathsf{tp}_2(x, \mathsf{s}(y)), y) \to \mathsf{tp}_2(\mathsf{s}(x), \mathsf{s}(y)), & \cdots \end{array} \right\}.$$

Here $\overline{\mathsf{mul}}$ and $\overline{\mathsf{add}}$ are inverse operators of multiplication and addition of natural numbers, respectively, and tp_2 is a constructor representing pairs of two terms.

The rewrite relations by rewrite rules having at least an extra variable are infinitely branching and cause non-termination. However, it was shown that *narrowing* reduction can simulate ground rewrite sequences of EV-TRSs if either the sequences are *EV-safe* or the systems are right-linear [10,11]. Here, a rewrite sequence on an EV-TRS is called *EV-safe* if any redex introduced by means of extra variables is not reduced along the sequence. Typical instances of EV-safe sequences are rewrite sequences where extra variables are substituted with normal forms at every rewrite step. Moreover, the EV-safety is not restrictive for inverse computation. Proving termination of narrowing sequences starting from ground terms is required because termination of inverse computation by EV-TRSs generated in [12,13] coincides with that of the narrowing sequences. Termination of such narrowing seems to be very similar to that of rewriting, while that of narrowing starting from arbitrary terms is not. For example, narrowing of R_1 starting from meaningful terms $\overline{\mathsf{mul}}(\mathsf{s}^n(\mathsf{0}))$ $(n \geq 0)$ is terminating, while it is not terminating if starting from arbitrary ground terms. The only way to prove the termination is mathematical induction on n by hand.

To prove termination of inverse computation, the *dependency pair method* [1] for proving termination of rewriting has already been extended to narrowing, by adding a condition. The extended method is applicable for constructor or right-linear systems [11]. However, the method is slightly too weak to prove termination of $\overline{\mathsf{mul}}(\mathsf{s}^n(\mathsf{0}))$.

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In this paper, we refine the dependency pair method extended in [11], that is, we remove the additional condition on the original method [1]. This refinement fills a gap between the methods for rewriting and narrowing starting from ground terms. We also extend the *dependency graph method* [1] for proving termination of rewriting, which is stronger than the dependency pair method, for proving termination of the narrowing. These methods are also applicable to either constructor or right-linear systems. One of the differences between the dependency graph methods for rewriting and narrowing is that an additional condition is imposed on argument filterings used in the method to maintain the groundness of sequences obtained by applying the argument filterings to dependency chains of narrowing.

A similar method for proving termination of narrowing was proposed by J. Christian [3]. This method corresponds to the well-known termination proof method that R is terminating if and only if there exists a reduction order $>$ such that $R \subseteq >$ [8], where the left-hand sides of rules are *flat*. At every step of narrowing of the TRSs, the number of kinds of variables does not increase. Therefore, the method for rewriting works well for narrowing. However, it seems to be difficult to apply this method to the termination proof of EV-TRSs, since the existence of extra variables impedes the main feature of this method.

This paper follows the basic notions of term rewriting [2,14], and the definition of *argument filterings* in [6]. *Extra variables* of a rewrite rule $l \to r$ are variables not in l but in r. A *TRS with extra variables* (*EV-TRS*, for short) is a set of rewrite rules that may have extra variables. The set of all variables in a term t is denoted by $\mathcal{V}ar(t)$. *Narrowing* of EV-TRSs is defined similarly to that of TRSs [5]. Let R be an EV-TRS. A term s is said to be *narrowable* into a term t at a non-variable occurrence p, written as $s \;{}_{\sigma|_{\mathcal{V}ar(s)}}\!\!\leadsto^{p}_{R}\; t$, if there exist a variant $\rho' : l \to r$ of a rewrite rule ρ in R and a substitution σ such that $t \equiv (s[r]_p)\sigma$ and σ is a most general unifier of $s|_p$ and l. The relation $\leadsto_R$ is called *narrowing* of R. A term that is reachable from a ground term is said to be *with a ground antecedent*. We simply say that narrowing on terms with ground antecedents is *narrowing with ground antecedents* (*GA-narrowing*, for short), that is, the binary relation $\underset{\mathrm{GA}}{\leadsto}{}_R = \{(s,t) \mid s \leadsto_R t, (\exists s_0 \in \mathcal{T}(\mathcal{F}), s_0 \overset{*}{\leadsto}_R s)\}$. For example, the set $R_2 = \{\ \mathsf{f}(x,\mathsf{0}) \to \mathsf{s}(x), \mathsf{g}(x) \to \mathsf{h}(x,y), \mathsf{h}(\mathsf{0},x) \to \mathsf{f}(x,x)\ \}$ is an EV-TRS. The ground term $\mathsf{g}(\mathsf{0})$ is narrowable by three steps to $\mathsf{s}(\mathsf{0})$, that is, $\mathsf{g}(\mathsf{0}) \leadsto_{R_2} \mathsf{h}(\mathsf{0},y) \leadsto_{R_2} \mathsf{f}(y,y) \;{}_{\{y\mapsto \mathsf{0}\}}\!\!\leadsto_{R_2} \mathsf{s}(\mathsf{0})$.

2 Dependency Pair Method for Narrowing

In this section, we refine the dependency pair method for narrowing [11].

Unlike the case of rewriting, the approach by dependency pairs is not applicable to all EV-TRSs [11]. It is applicable to the systems whose narrowing has the *TRAT property*, which is necessary to guarantee the existence of 'minimal' dependency chains for infinite sequences. Let R be an EV-TRS and $\to$ be a subset of either $\to_R$ or $\leadsto_R$. An infinite $\to$-sequence is said to be *almost terminating with respect to* $\to$ if every proper subterm of its initial term is terminating with respect to $\to$. An almost terminating $\to$-sequence is said to be *top-reduced* if it contains a reduction step at the root (top) position. An infinite $\to$-sequence *has a TRAT sequence* if there exists a subterm of its initial term, which causes a top-reduced almost terminating $\to$-sequence. The relation $\to$ has the *TRAT property* (*top-reduced almost terminating property*) if every infinite $\to$-sequence has a TRAT sequence. Unfortunately, narrowing does not generally have the TRAT property.

Example 1. Consider the TRS $R_3 = \{\ \mathsf{f}(\mathsf{f}(x)) \to x\ \}$ over a signature with a binary symbol c. We have the infinite $\leadsto_{R_3}$-sequence $\mathsf{c}(\mathsf{f}(x),x) \;{}_{\{x\mapsto \mathsf{f}(x')\}}\!\!\leadsto_{R_3} \mathsf{c}(x',\mathsf{f}(x')) \;{}_{\{x'\mapsto \mathsf{f}(x'')\}}\!\!\leadsto_{R_3}$

$\mathsf{c}(\mathsf{f}(x''), x'') \rightsquigarrow_{R_3} \cdots$. Since the above infinite almost-terminating sequence is not top-reduced, narrowing of R_3 does not have the TRAT property.

However, narrowing of EV-TRSs in some classes has the TRAT property.

Proposition 2 ([11]). *Let R be an EV-TRS.*

- *If R is right-linear, then $\rightsquigarrow_R$ has the TRAT property on linear terms.*
- *If R is a constructor system, then $\rightsquigarrow_R$ has the TRAT property.*

Dependency pairs of EV-TRSs are defined similarly to those of TRSs [1] (see [11]). The set of all dependency pairs of R is denoted by $\mathcal{DP}(R)$. *R-chains* are sequences of dependency pairs that are in turn connected with the rewrite relation of R. Chains for narrowing are defined similarly by connecting dependency pairs with the narrowing of R.

Definition 3 ([11]). Let R be an EV-TRS. A sequence $\langle s_1, t_1\rangle\ \langle s_2, t_2\rangle \cdots$ of dependency pairs of R is called an *R-narrowing-chain* if there exist terms $u_1, u_2, \cdots$ and substitutions $\sigma_1, \sigma_2, \cdots$ such that $u_1 \rightsquigarrow^{\varepsilon}_{\{s_1 \to t_1\}} t_1\sigma_1 \overset{*}{\rightsquigarrow}^{\varepsilon<}_{R} u_2 \rightsquigarrow^{\varepsilon}_{\{s_2 \to t_2\}} t_2\sigma_2 \overset{*}{\rightsquigarrow}^{\varepsilon<}_{R} \cdots$ where $\mathcal{V}ar(u_i) \cap \mathcal{V}ar(s_i, t_i) = \emptyset$ and σ_i is a most general unifier of s_i and u_i. In particular, the sequence is said to be *with a ground antecedent* if there exists a ground term s_0 such that $s_0 \overset{*}{\rightsquigarrow}^{\varepsilon<}_{R} u_1$.

To refine the method in [11], we prepare the following lemma.

Lemma 4. *Let R be an EV-TRS, $(\succsim, \succ)$ a reduction pair, π an argument filtering, s and t terms, and δ a substitution. If $s\ {}_{\delta}\overset{*}{\rightsquigarrow}_R\ t$ and $\pi(R) \subseteq \succsim$ (respectively $s\ {}_{\delta}\rightsquigarrow^{\varepsilon}_{\{l \to r\}}\ t$ and $l \succ r \subseteq \succsim$), then $\pi(s\delta) \succsim \pi(t)$ (respectively $\pi(s\delta) \succ \pi(t)$).*

The dependency pair method [1] for proving termination of rewriting was extended for GA-narrowing and narrowing in [11], and the extended method is refined as follows.

Theorem 5. *Let R be an EV-TRS, and suppose that $\rightsquigarrow_R$ has the TRAT property. R is terminating with respect to $\underset{\mathrm{GA}}{\rightsquigarrow}_R$ if and only if there exist a reduction pair $(\succsim, \succ)$ and an argument filtering π such that*

(a) $\pi(R) \subseteq \succsim$ and $\pi(\mathcal{DP}(R)) \subseteq \succ$.

R is terminating with respect to $\rightsquigarrow_R$ if $(\succsim, \succ)$ and π satisfy (a) and

(b) for all pairs $\langle s, t\rangle \in \mathcal{DP}(R)$, $\pi(t)$ is ground.

Proof (Sketch). By using Lemma 4 instead of Lemma 4.14 [11] in the proof of Theorem 4.16 [11], we can easily construct from an infinite R-narrowing-chain an infinite sequence on $\succ$. It follows from Theorem 4.9 in [11] that $\underset{\mathrm{GA}}{\rightsquigarrow}_R$ is not terminating. □

Condition (a) is the same as that in the case of rewriting. The refined point from the method in [11] is the removal of the restriction against argument filterings and the condition "$\pi(R) \cup \pi(\mathcal{DP}(R)) \subseteq \supseteq_{\mathcal{V}ar}$" where $\supseteq_{\mathcal{V}ar} = \{(s,t) \mid \mathcal{V}ar(s) \supseteq \mathcal{V}ar(t)\}$. Condition (b) guarantees that every infinite narrowing-chain not starting from any non-ground term has a postfix sequence that is a narrowing-chain with a ground antecedent. This guarantee reduces termination of narrowing to that of GA-narrowing.

Example 6. Consider the EV-TRS R_2 again. The dependency pairs of R_2 are $\langle \mathsf{g}^\sharp(x), \mathsf{h}^\sharp(x,y)\rangle$ and $\langle \mathsf{h}^\sharp(\mathsf{0}, x), \mathsf{f}^\sharp(x,x)\rangle$. Let π_2 be an argument filtering such that $\pi_2(\mathsf{f}) = \pi_2(\mathsf{g}) = \pi_2(\mathsf{h}) = \pi_2(\mathsf{f}^\sharp) = \pi_2(\mathsf{h}^\sharp) = [\,]$ and $\pi_2(\mathsf{g}^\sharp) = [1]$, and $>$ be a precedence over $\{\mathsf{f}, \mathsf{g}, \mathsf{h}, \mathsf{s}, \mathsf{0}, \mathsf{f}^\sharp, \mathsf{g}^\sharp, \mathsf{h}^\sharp\}$ such that $\mathsf{g}^\sharp > \mathsf{h}^\sharp > \mathsf{f}^\sharp \geq \mathsf{f} = \mathsf{g} = \mathsf{h} = \mathsf{s} = \mathsf{0}$. Then we have $\pi_2(R_2) \subseteq \geq_{\mathsf{lpo}}$ and $\pi_2(\mathcal{DP}(R_2)) \subseteq >_{\mathsf{lpo}}$, and hence $\underset{\mathrm{GA}}{\rightsquigarrow}_{R_2}$ is terminating. Moreover, $\rightsquigarrow_{R_2}$ is terminating because $\pi_2(\mathsf{h}^\sharp(\mathsf{0}, x))$ and $\pi_2(\mathsf{f}^\sharp(x,x))$ are ground.

The sufficient condition for termination of GA-narrowing in Theorem 5 corresponds to that for termination of rewriting. From this fact, the implementation of an automatic termination proof by Theorem 5 can be easily realized by relaxing the restriction "being TRSs" on inputs of existing tools, which are based on the dependency pair method for rewriting, to "being EV-TRSs". As we describe later, such implementation will be faulty if the existing tools are related to the dependency graph method.

3 Dependency Graph Method for Narrowing

As described in the previous section, we succeeded in extending the dependency pair method for rewriting to a method for GA-narrowing, without differences. This extension makes it also possible to extend the *dependency graph method* for rewriting [1] to a method for GA-narrowing. Unfortunately, unlike the case of the dependency pair method, we must add a condition to the original method. The dependency pair method cannot prove termination of the EV-TRS $R_4 = \{\mathsf{f}(x,x) \to y\}$, but the dependency graph method enables us to prove it.

We first define the dependency graph for narrowing.

Definition 7. Let R be an EV-TRS. The *narrowing dependency graph* of R, denoted by $\mathcal{NDG}(R)$, is the directed graph whose nodes are the dependency pairs of R, and there is an arc from $\langle s,t\rangle$ to $\langle u,v\rangle$ if $\langle s,t\rangle\langle u,v\rangle$ is an R-narrowing-chain.

The notion of *strongly connected subgraphs* (*SCSs*, for short) of $\mathcal{NDG}(R)$ is the same as the notion of 'cycle' in [14].

Theorem 8. *Let R be an EV-TRS. R is terminating with respect to $\underset{\text{GA}}{\leadsto}{}_R$ if for every SCS $\mathcal{P}$ in $\mathcal{NDG}(R)$ there exist a reduction pair $(\succsim, \succ)$ and an argument filtering π such that*

(a) $\pi(R) \subseteq \succsim$, $\pi(\mathcal{P}) \subseteq \succsim$ and $\pi(\mathcal{P}) \cap \succ \neq \emptyset$, and
(b) either $\pi(P') \subseteq \succsim$ or $\pi(R) \cup \pi(P') \subseteq \supseteq_{\mathcal{V}ar}$, where P' is a set of dependency pairs, each of which is reachable to a pair in $\mathcal{P}$ on the graph $\mathcal{NDG}(R)$.

R is terminating with respect to $\leadsto_R$ if for every SCS $\mathcal{P}$ in $\mathcal{NDG}(R)$, there exist a reduction pair $(\succsim, \succ)$ and an argument filtering π such that (a) and

(c) for a pair $\langle s,t\rangle$ in $\mathcal{P}$, $\pi(t)$ is ground.

Proof (Sketch). Similarly to the proof of this method for rewriting, we can easily show that an infinite narrowing-chain with a ground antecedent, which is caused by an SCS $\mathcal{P}$, implies an infinite sequence of $\succ$. To guarantee the existence of a ground antecedent for the sequence of $\succ$, condition (b) is used. Note that (c) implies (b). □

Condition (a) is similar to that in the case of rewriting, and (b) guarantees the existence of ground antecedents (with respect to $\succ$) for looping by $\mathcal{P}$. For example, consider the EV-TRS $R_5 = \{\ \mathsf{g}(x) \to \mathsf{add}(y,x),\ \mathsf{add}(\mathsf{0},y) \to y,\ \mathsf{add}(\mathsf{s}(x),y) \to \mathsf{s}(\mathsf{add}(x,y))\ \}$. It is clear that $\underset{\text{GA}}{\leadsto}{}_{R_5}$ is not terminating. If condition (b) in Theorem 8 is missing, then termination of $\underset{\text{GA}}{\leadsto}{}_{R_5}$ can be proved. To avoid such incorrect proof, condition (b) is necessary. Condition (b) makes it difficult to introduce this method to existing termination provers based on the dependency graph method. Condition (c) corresponds to Theorem 5 (b).

Example 9. Proving termination of $\rightsquigarrow_{R_2}$ and $\rightsquigarrow_{R_4}$ is easy by the dependency graph method because there is no SCS in their dependency graphs. Consider the EV-TRS $R_6 = \{\, \mathsf{g}(x) \to \mathsf{add}(x,y),\ \mathsf{add}(\mathsf{0},y) \to y,\ \mathsf{add}(\mathsf{s}(x),y) \to \mathsf{s}(\mathsf{add}(x,y))\,\}$. The only SCS in $\mathcal{NDG}(R_6)$ is $\{\langle \mathsf{add}^\sharp(\mathsf{s}(x),y), \mathsf{add}^\sharp(x,y)\rangle\}$. Let π_6 be an argument filtering such that $\pi_6(\mathsf{add}) = [1]$, and $>$ be a precedence such that $\mathsf{g} = \mathsf{add} = \mathsf{s} = \mathsf{0} = \mathsf{add}^\sharp$. Then we have $\pi_6(R_6) \subseteq \geq_{\mathsf{lpo}}$ and $\pi_6(\{\langle \mathsf{add}^\sharp(\mathsf{s}(x),y), \mathsf{add}^\sharp(x,y)\rangle\}) \subseteq \geq_{\mathsf{lpo}} \cap >_{\mathsf{lpo}}$. Therefore, $\underset{\mathrm{GA}}{\rightsquigarrow}_{R_6}$ is terminating.

As with dependency graphs for rewriting, narrowing dependency graphs are not computable in general because it is undecidable whether two dependency pairs form a narrowing-chain. However, the dependency graphs are approximations of the corresponding narrowing dependency graphs. Here, we denote the dependency graph of R by $\mathcal{DG}(R)$.

Theorem 10. *For every EV-TRS R, $\mathcal{NDG}(R)$ is a subgraph of $\mathcal{DG}(R)$.*

Note that the converse of the above theorem does not hold in general. The above theorem makes it possible for cases of narrowing to use several (computable) approximation techniques for dependency graphs, such as the *estimated dependency graphs* [1], *s-, nv- and growing-approximations* [9], *approximations based on ω- and Ω-reductions* [7], and so on, as approximations for narrowing ones.

We expect to introduce the notion of *usable rules* [4] to Theorem 8. Such an extended method will enable us to prove the termination of $\overline{\mathsf{mul}}(\mathsf{s}^n(\mathsf{0}))$ on R_1.

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