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Narrowing-Based Simulation of Term Rewriting Systems with Extra Variables and its Termination Proof

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Abstract

Term rewriting systems (TRSs) extended by allowing to contain extra variables in their rewrite rules are called EV-TRSs. They are ill-natured since every one-step reduction by their rules with extra variables is infinitely branching and they are not terminating. To solve these problems, this paper shows that narrowing can simulate reduction sequences of EV-TRSs as narrowing sequences starting from ground terms. We prove the soundness of ground narrowing sequences for the reduction sequences. We prove the completeness for the case of right-linear systems, and also for the case that any redex reduced in the reduction sequence is not introduced by means of extra variables. Moreover, we give a method to prove the termination of the simulation, extending the dependency pair method to prove termination of TRSs, into that of narrowing on EV-TRSs starting from ground terms. We show that the method is useful for right-linear or constructor systems.

1 Introduction

An *extra variable* is a variable appearing only in the right-hand side of a rewrite rule. Term rewriting systems (TRSs) extended by allowing to contain extra variables in their rewrite rules are called *EV-TRSs*. Especially they are *proper* if they contain at least one extra variable. Proper EV-TRSs are ill-natured since every one-step reduction by their rules with extra variables is infinitely

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branching even if we regard renamed terms as the same, and since none of them are terminating.

On the other hand, as a transformational approach to inverse computation, we have proposed an algorithm to generate a program computing the inverses of the functions defined by a given constructor TRS [15,16]. The generated programs are TRSs if the given TRSs are non-erasing. However, they are EV-TRSs in general. This fact gives rise to necessity of a method to simulate reductions of EV-TRSs.

This paper shows how to simulate reduction sequences of EV-TRSs, and also discusses how to prove termination of the simulation. We first show that narrowing [7] can simulate reduction sequences of EV-TRSs as narrowing sequences starting from ground terms. Such a simulation solves the infinitely-branchingness problem, and terminates for some EV-TRSs even if they are proper. We prove the soundness of the simulation. Then, we prove the completeness of the simulation in case of right-linear systems, and also in case that any redex reduced in the reduction sequences is not introduced by means of extra variables. One of the typical instances of the latter case is a sequence constructed by substituting normal forms for extra variables. As a technique to prove termination of the simulation, we extend the dependency pair method to prove termination of TRSs, which was proposed by T. Arts and J. Giesl [1], into that of narrowing on EV-TRSs, which starts from ground terms. We show that this technique is applicable to right-linear or constructor systems.

This paper is organized as follows. In Section 3, we explain the idea how to simulate the reduction of EV-TRSs using narrowing, and prove the soundness and completeness. In Section 4, we discuss the termination of the simulation, i.e., of narrowing starting from ground terms. Section 5 introduces related works. We give the proofs of some theorems and lemmas in the appendix.

2 Preparation

This paper follows standard notation of term rewriting [2,8,17]. In this section, we briefly describe notations used in this paper.

Let $\mathcal{F}$ be a *signature* and $\mathcal{X}$ be a countably infinite set of variables. The set of *terms* over $\mathcal{F}$ (and $\mathcal{X}$) is denoted by $T(\mathcal{F},\mathcal{X})$. The set $T(\mathcal{F},\emptyset)$ of *ground* terms is simply written as $T(\mathcal{F})$. For a function symbol f, $\texttt{arity}(f)$ denotes the number of arguments of f. The *identity* of terms s and t is denoted by $s \equiv t$. The set of variables in terms $t_1,\ldots,t_n$ is represented as $\mathcal{V}ar(t_1,\ldots,t_n)$. The *top* (or *root*) symbol of a term t is denoted by $\texttt{top}(t)$. If f is a unary function symbol, then $f^n(t)$ abbreviates the term $f(f(\cdots f(t)\cdots))$, the n-fold application of f to t.

We use $\mathcal{O}(t)$ to denote the set of all *positions* of term t, and $\mathcal{O}_{\mathcal{F}}(t)$ and $\mathcal{O}_{\mathcal{X}}(t)$ to denote the set of function-symbol and variable positions of t, respectively. We use ε to represent the top position. For $p,q \in \mathcal{O}(t)$, we write $p \leq q$ if there exists p' satisfying $pp' = q$. The *subterm* at a position $p \in \mathcal{O}(t)$ is represented by $t|_p$. A *context* C is a term with exactly one-hole $\Box$. $C[t]_p$ is a term obtained from C by replacing $\Box$ at p with a term t. The notation $u \trianglelefteq t$ means that u is a subterm of t.

A *substitution* is a mapping σ from $\mathcal{X}$ to $T(\mathcal{F}, \mathcal{X})$ such that $\sigma(x) \not\equiv x$ for finitely many $x \in \mathcal{X}$. We use σ, δ and θ to denote substitutions. Substitutions are naturally extended to mappings from $T(\mathcal{F}, \mathcal{X})$ to $T(\mathcal{F}, \mathcal{X})$ and $\sigma(t)$ is often written as $t\sigma$. We call $t\sigma$ an *instance* of t. The *composition* of σ and θ, denoted by $\sigma\theta$, is defined as $x\sigma\theta = \theta(\sigma(x))$. The *domain* and *range* of σ are defined as $\mathcal{D}om(\sigma) = \{x \in \mathcal{X} \mid x\sigma \not\equiv x\}$ and $\mathcal{R}an(\sigma) = \{x\sigma \mid x \in \mathcal{D}om(\sigma)\}$, respectively. The set of all variables occurring in $\mathcal{R}an(\sigma)$ is denoted by $\mathcal{VR}an(\sigma)$, i.e., $\mathcal{VR}an(\sigma) = \bigcup_{t \in \mathcal{R}an(\sigma)} \mathcal{V}ar(t)$. We call σ *ground* if $\mathcal{VR}an(\sigma) = \emptyset$. We write $\{\, x_1 \mapsto t_1, \ldots, x_n \mapsto t_n \,\}$ as σ if $\mathcal{D}om(\sigma) = \{x_1, \ldots, x_n\}$ and $x_i\sigma \equiv t_i$ for each i, and write $\emptyset$ instead of σ if $\mathcal{D}om(\sigma) = \emptyset$. We write $\sigma = \theta$ if $\mathcal{D}om(\sigma) = \mathcal{D}om(\theta)$ and $\sigma(x) \equiv \theta(x)$ for all $x \in \mathcal{D}om(\sigma)$. The *restriction* of σ to $X \subseteq \mathcal{X}$ is denoted by $\sigma|_X$, i.e., $\sigma|_X = \{\, x \mapsto \sigma(x) \mid x \in \mathcal{D}om(\sigma) \cap X \,\}$. We write $\sigma \lesssim \sigma'$ if there exists θ satisfying $\sigma\theta = \sigma'$.

A *rewrite rule* is a pair (l, r), written as $l \to r$, where l ($\notin \mathcal{X}$) and r are terms. It may have a unique label ρ and be written as $\rho : l \to r$. A variable appearing only in the right-hand side of a rule ρ is called an *extra variable*, and the set of all extra variables in ρ is denoted by $\mathcal{EV}ar(\rho)$. An *EV-TRS* is a finite set R of rewrite rules. Especially, it is called a *term rewriting system* (TRS, for short) if every rewrite rule $l \to r \in R$ satisfies $\mathcal{V}ar(l) \supseteq \mathcal{V}ar(r)$. The *reduction relation* $\to_R$ is a binary relation on terms defined by $\to_R = \{\, (C[l\sigma]_p, C[r\sigma]_p) \mid C \text{ is a context}, l \to r \in R \,\}$. When we explicitly specify the position p and the rule ρ in $s \to_R t$, we write $s \to_R^{[p,\rho]} t$ or $s \to_R^{p} t$. As usual $\xrightarrow{*}_R$ and $\xrightarrow{n}_R$ are a reflexive and transitive closure of $\to_R$, and the n-step reduction of $\to_R$, respectively. We call $s_0 \to_R s_1 \to_R \cdots$ a *reduction sequence* of R. A term t is a *normal form* if there is no term u such that $t \to_R u$.

Let R be an EV-TRSs over a signature $\mathcal{F}$. The set of *defined symbols* of R is defined as $\mathcal{D}_R = \{\, \mathtt{top}(l) \mid l \to r \in R \,\}$, and the set of *constructors* of R as $\mathcal{C}_R = \mathcal{F} \setminus \mathcal{D}_R$. R is said to be a *constructor system* if every rule $f(t_1, \ldots, t_n) \to r \in R$ satisfies that $t_i \in T(\mathcal{C}_R, \mathcal{X})$ for all i.

Terms s and t are *renamings* (or *variants*) if s and t are instances of each other. Letting $\to$ be a binary relation on terms, $\to$ is *finitely branching* if a set $\{\, t \mid s \to t \,\}$ is finite up to renaming for any term s. Otherwise, it is *infinitely branching*. Let R be an EV-TRS. If R is a TRS then $\to_R$ is finitely branching. Otherwise, however, it is infinitely branching in general.

Example 2.1 The following R_1 is an EV-TRS:

$$R_1 = \{\, f(x, 0) \to s(x),\ \ g(x) \to h(x, y),\ \ h(0, x) \to f(x, x),\ \ a \to b \,\}.$$

A term $g(0)$ can be reduced by the second rule above to any of terms, such as $h(0, 0)$, $h(0, g(0))$, $h(0, f(0, 0))$ and so on. Thus, $\to_{R_1}$ is infinitely branching.

$$
\begin{array}{llllllll}
\text{(i)} & & \vdots & & \vdots & \iddots & & \\
 & & \nearrow_{R_1} & h(0,a) & \to_{R_1} & \cdots & & \\
 & g(0) & \to_{R_1} & h(0,0) & \to_{R_1} & f(0,0) & \to_{R_1} & s(0) \\
 & & \searrow_{R_1} & h(0,g(a)) & \to_{R_1} & \cdots & & \\
 & & \vdots & & \vdots & \ddots & & \\
\text{(ii)} & g(0) & \rightsquigarrow_{R_1} & h(0,z) & \rightsquigarrow_{R_1} & f(z,z) & {}_{\{z\mapsto 0\}}\rightsquigarrow_{R_1} & s(0)
\end{array}
$$

Fig. 1. (i) Reduction, and (ii) narrowing sequences, starting from $g(0)$.

3 Simulating Reduction of EV-TRSs Using Narrowing

In this section, we discuss how to simulate reduction sequences of EV-TRSs. We first explain the idea intuitively. Then, we prove the soundness of the simulation, and also show conditions on which the simulation is complete.

In reduction sequences, arbitrary terms can be substituted for extra variables. For example, $g(0)$ is reduced to $h(0,t)$ for every term t by R_1 in Example 2.1, as shown in Fig. 1 (i). By using a fresh variable z as a representation of all t's and by unsing the narrowing [7] instead of the reduction, these reduction sequences in Fig. 1 (i) are represented by a single narrowing sequence in Fig. 1 (ii). This is the idea of the simulation.

A *unifier* of terms s and t is a pair (σ,σ') of substitutions such that $s\sigma \equiv t\sigma'$ [4]. A *most general unifier* of s and t, denoted by $\mathtt{mgu}(s,t)$, is a unifier (σ,σ') of s and t such that $\sigma \lesssim \theta$ and $\sigma' \lesssim \theta'$ for all unifiers (θ,θ') of s and t. It is known that a most general unifier is unique up to renaming if it exists.

The definition of narrowing on EV-TRSs follows the common definition of the narrowing [7] as follows:

Definition 3.1 Let R be an EV-TRS. A term s is said to be *narrowable* into a term t with a substitution δ, a position $p \in \mathcal{O}(s)$ and a rewrite rule $\rho \in R$, written as $s \; {}_{\delta}\!\rightsquigarrow_R^{[p,\rho]} t$, if there exist a context C, a term u and a substitution σ such that $p \in \mathcal{O}_{\mathcal{F}}(s)$, $s \equiv C[u]_p$, $t \equiv C\delta[r\sigma]_p$, $(\delta,\sigma) = \mathtt{mgu}(s, C[l]_p)$ [5] and $\mathcal{VR}an(\sigma|_{\mathcal{EV}ar(\rho)}) \cap \mathcal{V}ar(C\delta) = \emptyset$, where $\rho : l \to r$. We call $\rightsquigarrow_R$ *narrowing* by R. Note that δ, p and ρ may be omitted like as $s \rightsquigarrow_R t$, $s \rightsquigarrow_R^p t$ or $s \rightsquigarrow_R^{[p,\rho]} t$.

Note that each extra variable introduces a fresh variable, which does not

[4] Usually, an unifier θ of s and t is defined as $s\theta = t\theta$. If variables in s and t are disjoint, which is satisfied in the definition of the narrowing, these definitions of unifier are equivalent: $\theta = \sigma \cup \sigma'$. We use the pair representation to eliminate renaming variables of rewrite rules in the definition of narrowing and to simplify treatments of variables in the proofs of theorems.

[5] This condition guarantees $\mathcal{VR}an(\delta) \cap (\mathcal{V}ar(s) \setminus \mathcal{D}om(\delta)) = \emptyset$ and $\mathcal{VR}an(\sigma) \cap (\mathcal{V}ar(s) \setminus \mathcal{D}om(\delta)) = \emptyset$ that are assumed implicitly in the common definition of narrowing behind the definition of the most general unifiers.

$$
\begin{array}{lll}
R_2 = \{ & add^\#(y) \to tp_2(0,y), & add^\#(s(z)) \to u_1(add^\#(z)), \\
& u_1(tp_2(x,y)) \to tp_2(s(x),y), & add^\#(add(x,y)) \to tp_2(x,y), \\
& mul^\#(0) \to tp_2(0,y), & mul^\#(0) \to tp_2(x,0), \\
& mul^\#(s(z)) \to u_2(add^\#(z)), & u_2(tp_2(w,y)) \to u_3(mul^\#(w),y), \\
& u_3(tp_2(x,s(y)),y) \to tp_2(s(x),s(y)), & mul^\#(mul(x,y)) \to tp_2(x,y) \quad \}
\end{array}
$$

Fig. 2. The EV-TRS computing the inverses of addition and multiplication.

occur in $\mathcal{V}ar(s\delta)$ and $\mathcal{VR}an(\sigma|_{\mathcal{V}ar(l)})$, since the definition of the most general unifiers guarantees that $x\sigma \in \mathcal{X} \setminus (\mathcal{VR}an(\delta) \cup \mathcal{VR}an(\sigma))$ for all $x \in \mathcal{EV}ar(\rho)$, and that $x \not\equiv y$ implies $x\sigma \not\equiv y\sigma$ for any $x, y \in \mathcal{EV}ar(\rho)$. We write $s \;{}_\delta\!\overset{n}{\leadsto}_R\; t$ or $s \;{}_\delta\!\overset{*}{\leadsto}_R\; t$ if there exists a narrowing derivation $s \equiv t_0 \;{}_{\delta_0}\!\leadsto_R\; t_1 \;{}_{\delta_1}\!\leadsto_R \cdots {}_{\delta_{n-1}}\!\leadsto_R\; t_n \equiv t$, called a *narrowing sequence*, where $\delta = \delta_0\delta_1 \cdots \delta_{n-1}$ and if $n = 0$ then $\delta = \emptyset$. Note that δ may be omitted like as $s \overset{n}{\leadsto}_R t$ or $s \overset{*}{\leadsto}_R t$. Especially, a narrowing sequence is said to be *ground* if it starts from a ground term. From the definition of narrowing, it is clear that $\leadsto_R$ are finitely branching for every EV-TRS R, and also that $\to_R = \leadsto_R$ on ground terms for every TRS R.

Here, we show an example of the simulation by narrowing on EV-TRSs.

Example 3.2 The system computing inverse images $add^\#$ of addition and $mul^\#$ of multiplication on natural numbers is resulted in the EV-TRS seen in Fig. 2 [15]. Considering the narrowing sequences starting from $mul^\#(s^4(0))$, there exist only 16 finite-paths up to renaming. This means that all solutions of $mul^\#(s^4(0))$ are found in finite time and space. One of such paths is as follows:

$$
\begin{aligned}
mul^\#(s^4(0)) &\overset{*}{\leadsto}_{R_2} u_3(u_3(mul^\#(0), s(0)), s(0)) \\
&\leadsto_{R_2} u_3(u_3(tp_2(0,y), s(0)), s(0)) \\
&{}_{\{y \mapsto s^2(0)\}}\!\leadsto_{R_2} u_3(tp_2(s(0), s^2(0)), s(0)) \leadsto_{R_2} tp_2(s^2(0), s^2(0)).
\end{aligned}
$$

The following theorem shows the soundness whose proof is similar to that on TRSs [7].

Theorem 3.3 *Let R be an EV-TRS. For all ground terms s, all terms t and all substitution δ, $s \;{}_\delta\!\overset{*}{\leadsto}_R\; t$ implies $s\delta \overset{*}{\to}_R t$.*

Proof. We prove by induction on n that $s \;{}_\delta\!\overset{n}{\leadsto}_R\; t$ implies $s\delta \overset{*}{\to}_R t$.

Since the case of $n = 0$ is trivial, we assume that $s \;{}_\delta\!\overset{n-1}{\leadsto}_R\; u \equiv C[u']_p \;{}_{\delta'}\!\leadsto_R\; C\delta'[r\sigma]_p \equiv t$, where $\rho : l \to r \in R$, and $(\delta', \sigma) = \mathtt{mgu}(u, C[l]_p)$. From the definition of the most general unifiers, we have $u'\delta' \equiv l\sigma$. By induction hypothesis, we have $s\delta \overset{*}{\to}_R u$. Then, it follows from the stability of reduction that $s\delta\delta' \overset{*}{\to}_R u\delta'$. Therefore, we have $s\delta\delta' \overset{*}{\to}_R u\delta' \equiv (C[u']_p)\delta' \equiv C\delta'[l\sigma]_p \to_R C\delta'[r\sigma]_p \equiv t$. □

Now we introduce the notion of EV-safe reduction sequences. We say that a reduction sequence is *EV-safe* if any redex, which is reduced in the sequence, is not introduced by means of extra variables. A precise definition of this notion is found in Appendix A. A typical instance of EV-safe reductions is a reduction sequence of which a normal form is substituted for each extra variable in every step.

The following theorems are results on the completeness. These proofs are given in Appendix B.

Theorem 3.4 *Let R be an EV-TRS. Let s and t be ground terms. If $s \xrightarrow{*}_R t$ is EV-safe then there exist a term t' and substitution θ such that $s \overset{*}{\rightsquigarrow}_R t'$ and $t \equiv t'\theta$.*

Theorem 3.5 *Let R be a right-linear EV-TRS. Let s and t be ground terms. Then, $s \xrightarrow{*}_R t$ implies $s \overset{*}{\rightsquigarrow}_R t'$ and $t \equiv t'\theta$ for some linear term t' and substitution θ.*

Theorem 3.4 is a more general variant of the completeness lemma of narrowing [7].

The soundness and completeness in the above theorems guarantee only on reduction sequences starting from a ground term. However, they are sufficient because variables in reduction sequences can be regarded as fresh constants.

The following example shows that the completeness does not hold in general, and also shows that the conditions in Theorem 3.4 and 3.5 are essential and necessary.

Example 3.6 Consider the sequence starting from $g(0)$ by R_1 in Example 2.1 again. We have the non-EV-safe reduction-sequence $g(0) \to_{R_1} h(0, a) \to_{R_1} f(a, a) \to_{R_1} f(a, b)$. This sequence cannot be simulated by narrowing. In fact, $g(0)$ cannot be narrowable to any term having the instance $f(a, b)$.

4 Termination Proof of The Simulation

In order to prove termination of ground narrowing-sequences, we extend the dependency pair method [1] into that of narrowing, especially, which starts from ground terms. We show that this method is useful for right-linear or constructor systems.

Let $\to$ be a binary relation on terms. We say that $\to$ is *monotone* if $s \to t$ implies $C[s] \to C[t]$ for all contexts C. A term t is said to be *strongly normalizing* with respect to $\to$ ($SN_t^{\to}$, for short) if there is no infinite sequence $t \to t_1 \to \cdots$. We say that $\to$ is *strongly normalizing* ($SN^{\to}$) if $SN_t^{\to}$ holds for every term t, and it is *ground strongly normalizing* ($GSN^{\to}$) if $SN_t^{\to}$ holds for every ground term t. It is trivial that $SN^{\to_R}$ is equivalent to $GSN^{\to_R}$ if R is a TRS. Since the proposed simulation is done by narrowing sequences starting from ground terms, the property $GSN^{\rightsquigarrow_R}$ is enough for the termination of the simulation. The following proposition holds obviously because ground narrowing and reduction sequences on TRSs are equivalent.

Proposition 4.1 *Let R be a TRS. Then, $SN^{\to_R}$ if and only if $GSN^{\rightsquigarrow_R}$.*

The following claim associated with $SN^{\leadsto_R}$ and $GSN^{\leadsto_R}$ holds obviously.

Proposition 4.2 *Let R be an EV-TRS. Then, $SN^{\leadsto_R}$ implies $GSN^{\leadsto_R}$.*

The converse of the above does not hold. For example, considering a TRS $R_3 = \{\, d(0) \to 0,\, d(s(x)) \to s(s(d(x)))\,\}$, $GSN^{\leadsto_{R_3}}$ holds but $SN^{\leadsto_{R_3}}$ does not hold. Thus, since $SN^{\leadsto_R}$ does not hold for most of EV-TRSs (even TRSs), the results on $SN^{\leadsto_R}$ have very restrictive power.

4.1 Top Reduced Almost Terminating Property

Here, we introduce the *top reduced almost terminating* property. Let R be an EV-TRS, and $\to$ be a relation either $\to_R$ or $\leadsto_R$. An infinite sequence $t \to t_1 \to \cdots$ is said to be *almost terminating* if $SN^{\to}_u$ for every proper subterm u of t. An almost terminating sequence is said to be *top reduced* if it contains $\to^{\varepsilon}$. We say that $\to$ has *top reduced almost terminating* (TRAT, for short) property if there exists a top reduced almost-terminating sequence starting from a subterm u of t for every non-terminating term t, that is, $\neg SN^{\to}_t$. The following clearly holds:

Proposition 4.3 *If $\to$ is monotone then $\to$ has TRAT property.*

It is known that $\to_R$ is monotone. However, $\leadsto_R$ is not monotone in general, and hence $\leadsto_R$ does not have TRAT property in general. For example, a TRS $R_4 = \{\ f(f(x)) \to x\ \}$ does not have TRAT property. Because the following almost terminating sequence is not top reduced:

$$c(f(x), x)\ {}_{\{x \mapsto f(x')\}}\!\leadsto_{R_4} c(x', f(x'))\ {}_{\{x' \mapsto f(x'')\}}\!\leadsto_{R_4} c(f(x''), x'')\ {}_{\{x' \mapsto f(x'')\}}\!\leadsto_{R_4} \cdots.$$

For right-linear systems, $\leadsto_R$ on linear terms has nice properties [14].

Proposition 4.4 *Let R be a right-linear EV-TRS.*

(i) *If $s \overset{*}{\leadsto}_R t$ for a linear term s, then t is linear.*

(ii) *The narrowing $\leadsto_R$ on linear terms is monotone.*

Proof. (i) We consider only the case of one-step; $C[u]_p\ {}_{\delta}\!\leadsto^{[p,\rho]}_R C\delta[r\sigma]_p$ where $C[u]_p$ is linear, $(\delta, \sigma) = \mathtt{mgu}(C[u]_p, C[l]_p)$, and $\rho : l \to r \in R$. If follows from the linearity of $C[u]_p$ that C and u are linear and $\mathcal{V}ar(C) \cap \mathcal{V}ar(u) = \emptyset$. From the definition of the most general unifiers, we can assume that $\mathcal{V}ar(C) \cap \mathcal{D}om(\delta) = \emptyset$. Hence, $C\delta \equiv C$. It follows from the linearity of u that every terms in $\mathcal{R}an(\sigma)$ is linear. Then, since r is linear, $r\sigma$ is also linear. Therefore, $C\delta[r\sigma]_p$ ($\equiv C[r\sigma]_p$) is linear.

(ii) This claim follows obviously from (i). □

We also have a nice property on constructor systems. The proof of the following theorem is included in Appendix C.

$$
\begin{array}{cccccccc}
 & & \langle\, s_1,\ t_1 \,\rangle & & \langle\, s_2,\ t_2 \,\rangle & & \langle\, s_3,\ t_3 \,\rangle & \cdots \\
 & {\scriptstyle (\delta_1,\sigma_1) = \atop \texttt{mgu}(s'_1, s_1)} \vdots & \vdots & {\scriptstyle (\delta_2,\sigma_2) = \atop \texttt{mgu}(s'_2, s_2)} \vdots & \vdots & {\scriptstyle (\delta_3,\sigma_3) = \atop \texttt{mgu}(s'_3, s_3)} \vdots & \vdots & \\
\big(T(\mathcal{F} \cup \mathcal{G}) \ni \exists s_0 \overset{*}{\leadsto}_R \big) & s'_1 & t_1\sigma_1 \overset{*}{\leadsto}_R & s'_2 & t_2\sigma_2 \overset{*}{\leadsto}_R & s'_3 & t_3\sigma_3 \overset{*}{\leadsto}_R & \cdots
\end{array}
$$

Fig. 3. A (ground) $\langle\!\langle \leadsto_R, S \rangle\!\rangle$-chain.

Theorem 4.5 *Let R be a constructor EV-TRS. Then, $\leadsto_R$ has TRAT property.*

4.2 Dependency Pairs and Chains

The definition of dependency pairs of EV-TRSs is the same with that of TRSs [1]. To illustrate it, we prepare a fresh function symbol F not in a signature $\mathcal{F}$ for every defined symbol f. We call F the *capital symbol* of f. This paper uses small letters for function symbols in $\mathcal{F}$ and uses the string obtained by replacing the first letter of a defined symbol with the corresponding capital letter. For example, we use *Abc* as the capital symbol of a defined symbol *abc*. The set $\overline{\mathcal{D}}_R$ of all capital symbols determined by the set $\mathcal{D}_R$ of the defined symbols of R, is defined as $\overline{\mathcal{D}}_R = \{ F \mid f \in \mathcal{D}_R \}$. Moreover, we define $\overline{\mathcal{F}} = \mathcal{F} \cup \overline{\mathcal{D}}_R$.

Definition 4.6 Let R be an EV-TRS. The pair $\langle F(s_1, \ldots, s_n), G(t_1, \ldots, t_m) \rangle$ is called a *dependency pair* of R if there are a rewrite rule $f(s_1, \ldots, s_n) \to r \in R$ and a subterm $g(t_1, \ldots, t_m) \trianglelefteq r$ with $g \in \mathcal{D}_R$. The set of all dependency pairs of R is denoted by $\mathcal{DP}_R$.

Let $\langle s, t \rangle \in \mathcal{DP}_R$. We also call variables only in t (i.e.,in $\mathcal{V}ar(t) \setminus \mathcal{V}ar(s)$) extra variables, and write $\mathcal{EV}ar(\langle s, t \rangle)$ as the set of all extra variables of $\langle s, t \rangle$.

Next we define R-chains [1] and extend them to those constructed by narrowing.

Definition 4.7 Let R be an EV-TRS over a signature $\mathcal{F}$ and S be a set of pairs of terms over $\mathcal{F}$ and a signature $\mathcal{G}$. The sequence $\langle s_1, t_1 \rangle \langle s_2, t_2 \rangle \cdots$ of pairs in S is

(i) is called a $\langle\!\langle \to_R, S \rangle\!\rangle$*-chain* if there exists a substitution σ_i for every $i > 0$, such that $t_i\sigma_i \overset{*}{\to}_R s_{i+1}\sigma_{i+1}$, and

(ii) is called a $\langle\!\langle \leadsto_R, S \rangle\!\rangle$*-chain* if there exists a term s'_i and the most general unifier $(\delta_i, \sigma_i) = \texttt{mgu}(s'_i, s_i)$ for every $i > 0$, such that $t_i\sigma_i \overset{*}{\leadsto}_R s'_{i+1}$ (Fig. 3).

In the case of (ii), it is said to be *ground*, written as $s_0\langle s_1, t_1 \rangle \langle s_2, t_2 \rangle \cdots$, if there exists some ground term $s_0 \in T(\mathcal{F} \cup \mathcal{G})$ such that $s_0 \overset{*}{\leadsto}_R s'_1$. Note that a $\langle\!\langle \to_R, \mathcal{DP}_R \rangle\!\rangle$-chain is simply called an *R-chain* [1].

The following theorem shows the relationship between non-existence of infinite chains and termination of TRSs.

Theorem 4.8 ([1]) *Let R be a TRS. Then, $SN^{\to_R}$ if and only if there is no infinite R-chain.*

Then, we extend the above theorem to that of the narrowing. The proof of the following theorem is given in Appendix C.

Theorem 4.9 *Let R be an EV-TRS and $\leadsto_R$ has TRAT property.*

(i) *$SN^{\leadsto_R}$ if and only if there is no infinite $\langle\langle \leadsto_R, \mathcal{DP}_R \rangle\rangle$-chain.*

(ii) *$GSN^{\leadsto_R}$ if and only if there is no infinite ground $\langle\langle \leadsto_R, \mathcal{DP}_R \rangle\rangle$-chain.*

4.3 Eliminating All Extra Variables Using Argument Filtering

In the case of TRSs, the termination proof can be done by finding a reduction ordering to ensure no infinite chain. To find such an ordering, argument filtering functions [10] are known to be useful.

Definition 4.10 Let $\mathcal{G}$ be a signature. An *argument filtering* (AF, for short) is a function π such that for any $f \in \mathcal{G}$, $\pi(f)$ is either an integer i or a list $[i_1, \ldots, i_m]$ of integers where $n = \mathtt{arity}(f)$, $1 \leq i \leq n$ $1 \leq m \leq n$ and $1 \leq i_1 < \cdots < i_m \leq n$. Note that we assume $\pi(f) = [1, \ldots, n]$ if $\pi(f)$ is not defined explicitly. We can naturally extend π over terms as follows:

- $\pi(x) = x$ where $x \in \mathcal{X}$,
- $\pi(f(t_1, \ldots, t_n)) = \pi(t_i)$ where $\pi(f) = i$, and
- $\pi(f(t_1, \ldots, t_n)) = f(\pi(t_{i_1}), \ldots, \pi(t_{i_m}))$ where $\pi(f) = [i_1, \ldots, i_m]$.

Moreover, π is extended over a set S of term pairs as $\pi(S) = \{\ (\pi(s), \pi(t)) \mid (s,t) \in S\ \}$.

This paper assumes that $\pi(f)$ is not an integer but in form $[i_1, \ldots, i_m]$ for every defined symbol f, and also for every capital symbol. We say that such an AF function is *simple*.

The definitions of orderings in this paper follows those in [17]. A *quasi-ordering* $\succsim$ is a reflexive and transitive relation, its *strict* part $\succ$ is a partial ordering $\succ^{\mathrm{s}}$ defined as, $s \succ^{\mathrm{s}} t$ if and only if $s \succsim t$ and $t \not\succsim s$, and its *stable-strict* part $\succ^{\mathrm{ss}}$ is defined as, $s \succ^{\mathrm{ss}} t$ if and only if $s\sigma \succ^{\mathrm{s}} t\sigma$ for all ground substitutions σ. It is called *well-founded* if its stable-strict part is well-founded. A *quasi-reduction* ordering is a well-founded quasi-ordering $\succsim$ which is closed under contexts and substitutions.

Definition 4.11 The ordering $\succsim_\pi$ determined by a quasi-ordering $\succsim$ and an AF π is defined as follows:

- $s \succsim_\pi t$ if and only if $\pi(s) \succ \pi(t)$ or $\pi(s) \equiv \pi(t)$, and
- $s \succ_\pi t$ if and only if there are a contest C such that $\pi(s) \succ C[\pi(t)]$, or $\pi(s) \equiv C[\pi(t)]_p$ where $\varepsilon < p$.

It is known that $\succsim_\pi$ above is a quasi-reduction ordering [1,10,17].

To guarantee that no infinite chain exists, the following theorem is used.

Theorem 4.12 ([1]) *Let R be a TRS. There is no infinite R-chain if and only if there is an AF function π and a quasi-reduction ordering $\succsim$ such that*

- $l \succsim_\pi r$ *for every rule* $l \to r \in R$*, and*
- $s \succ_\pi t$ *for every dependency pair* $\langle s, t\rangle \in \mathcal{DP}_R$*.*

However, extra variables make it difficult to find such an ordering on proper EV-TRSs. Hence, we use AF functions again to eliminate all extra variables. Let R be an EV-TRS and π be an AF. We say that π *eliminates all extra variables* of R and $\mathcal{DP}_R$ if $\mathcal{V}ar(\pi(l)) \supseteq \mathcal{V}ar(\pi(r))$ for all rules $l \to r \in R$ and $\mathcal{V}ar(\pi(s)) \supseteq \mathcal{V}ar(\pi(t))$ for all dependency pairs $\langle s, t\rangle \in \mathcal{DP}_R$.

Proposition 4.13 *Let R be an EV-TRS, S be a set of term pairs and π be a simple AF that eliminates all extra variables of R and S.*

(i) *A ground $\langle\!\langle \leadsto_{\pi(R)}, \pi(S)\rangle\!\rangle$-chain is a $\langle\!\langle \to_{\pi(R)}, \pi(S)\rangle\!\rangle$-chain.*

(ii) *For any $\langle\!\langle \leadsto_{\pi(R)}, \pi(S)\rangle\!\rangle$-chain $\langle \pi(s_1), \pi(t_1)\rangle\ \langle \pi(s_2), \pi(t_2)\rangle \cdots$, if there exists a pair $\langle s_i, t_i\rangle \in S$ $(i > 0)$ such that $\pi(t_i)$ is ground, then there exists a $\langle\!\langle \to_{\pi(R)}, \pi(S)\rangle\!\rangle$-chain $\langle \pi(s_{i+1}), \pi(t_{i+1})\rangle\ \langle \pi(s_{i+2}), \pi(t_{i+2})\rangle \cdots$.*

The proof of the following lemma is found in Appendix C.

Lemma 4.14 *Let R be an EV-TRS, π be a simple AF function that eliminates all extra variables of R and $\mathcal{DP}_R$. Let s and t be terms such that $\pi(s)$ is ground. Then, $s \stackrel{*}{\leadsto}_R t$ implies $\pi(s) \stackrel{*}{\to}_{\pi(R)} \pi(t)$.*

Note that $\pi(s) \stackrel{*}{\leadsto}_{\pi(R)} \pi(t)$ if and only if $\pi(s) \stackrel{*}{\to}_{\pi(R)} \pi(t)$ since $\pi(R)$ is a TRS and $\pi(s)$ is ground.

No infinite $\langle\!\langle \leadsto_{\pi(R)}, \pi(\mathcal{DP}_R)\rangle\!\rangle$-chain gives us the following theorem.

Theorem 4.15 *Let R be an EV-TRS and π be a simple AF that eliminates all extra variables of R and $\mathcal{DP}_R$. Whenever there exists no infinite $\langle\!\langle \to_{\pi(R)}, \pi(\mathcal{DP}_R)\rangle\!\rangle$-chain then*

(i) *there exists no infinite ground $\langle\!\langle \leadsto_R, \mathcal{DP}_R\rangle\!\rangle$-chain, and*

(ii) *there exists no infinite $\langle\!\langle \leadsto_R, \mathcal{DP}_R\rangle\!\rangle$-chain if $\pi(t)$ is ground for all pairs $\langle s, t\rangle \in \mathcal{DP}_R$.*

Proof. (i) is easily proved by constructing an infinite $\langle\!\langle \to_{\pi(R)}, \pi(\mathcal{DP}_R)\rangle\!\rangle$-chain from an infinite ground $\langle\!\langle \leadsto_R, \mathcal{DP}_R\rangle\!\rangle$-chain, using Lemma 4.14 and Proposition 4.13 (i). The claim (ii) is proved from Proposition 4.13 (ii) and (i) of this theorem. □

If there is no extra variable in $\pi(R)$ and $\pi(\mathcal{DP}_R)$, we can check whether an infinite $\langle\!\langle \leadsto_{\pi(R)}, \pi(\mathcal{DP}_R)\rangle\!\rangle$-chain exists, by using the termination proof techniques [1,2,17], which is used to prove $SN^{\to_R}$ of a TRS R.

4.4 Termination Proof Theorem

We conclude from Theorem 4.9, 4.12 and 4.15 as follows:

Theorem 4.16 *Let R be an EV-TRS and $\rightsquigarrow_R$ has TRAT property. $GSN^{\rightsquigarrow_R}$ holds if there exist a quasi-reduction ordering $\gtrsim$ and a simple AF π that eliminates all extra variables of R and $\mathcal{DP}_R$, which satisfy both of the followings:*

(i) *$l \gtrsim_\pi r$ for every rule $l \to r \in R$, and*

(ii) *$s \succ_\pi t$ for every dependency pair $\langle s,t\rangle \in \mathcal{DP}_R$.*

Moreover, in the above case, $SN^{\rightsquigarrow_R}$ if (iii) *$\pi(t)$ is ground for all $\langle s,t\rangle \in \mathcal{DP}_R$.*

Proof. Assume that R is not $GSN^{\rightsquigarrow_R}$. Then, from the contraposition of Theorem 4.9 and 4.15, there exists an infinite $\langle\!\langle \rightsquigarrow_{\pi(R)}, \pi(\mathcal{DP}_R)\rangle\!\rangle$-chain. On the other hand, let a TRS $R' = \{\ l \to r\sigma|_{\mathcal{EV}ar(\rho)} \mid \rho : l \to r \in R, (\forall x \in \mathcal{EV}ar(\rho), x\sigma \in T(\mathcal{C}_R))\ \}$, we have $\pi(R) = \pi(R')$ and $\pi(\mathcal{DP}_R) = \pi(\mathcal{DP}_{R'})$. Moreover, $SN^{\to_{R'}}$ from Theorem 4.12. However, there exists an infinite $\langle\!\langle \to_R, \mathcal{DP}_R\rangle\!\rangle$-chain from Proposition 4.13. Therefore, contradiction. □

Note that Theorem 4.16 is also usable to check whether $SN^{\rightsquigarrow_R}$ of a TRS R holds, although the result is very restrictive because narrowing sequences seldom terminate. For example, even a simple TRS $\{\ f(s(x)) \to f(x)\ \}$ which terminates is not $SN^{\rightsquigarrow_R}$, since term $f(y)$ with variable y leads to an infinite narrowing sequence.

Remark that we can easily modify Theorem 4.16 (ii) and (iii) stronger by using dependency graphs [1]; the condition (ii) is that for every cycle $\mathcal{P}$ of dependency pairs, (a) $s \gtrsim_\pi t$ for every $\langle s,t\rangle \in \mathcal{P}$, and (b) $s \succ_\pi t$ for at least one $\langle s,t\rangle \in \mathcal{P}$; the condition (iii) is that for every cycle $\mathcal{P}$, $\pi(t)$ is ground for at least one $\langle s,t\rangle \in \mathcal{P}$. Such an extended theorem is more powerful.

Example 4.17 Consider R_1 in Example 2.1 again. The set of its dependency pairs is $\mathcal{DP}_{R_1} = \{\langle G(x), H(x,y)\rangle, \langle H(0,x), F(x,x)\rangle\}$. The dependency pair sequence $\langle G(x), H(x,y)\rangle\langle H(0,x), F(x,x)\rangle$ is an $\langle\!\langle \rightsquigarrow_{R_1}, \mathcal{DP}_R\rangle\!\rangle$-chain and $G(0)\langle G(x), H(x,y)\rangle\langle H(0,x), F(x,x)\rangle$ is a ground one.

Let π_1 be a simple AF function with

$$\pi_1(h) = \pi_1(H) = \pi_1(s) = \pi_1(g) = \pi_1(G) = \pi_1(f) = \pi_1(F) = [\ ].$$

Then, we have $\pi_1(R_1) = \{\ f \to s,\ g \to h,\ h \to f,\ a \to b\ \}$ and $\pi_1(\mathcal{DP}_{R_1}) = \{\langle G,H\rangle, \langle H,F\rangle\}$. It is clear that no infinite $\langle\!\langle \to_{\pi_1(R_1)}, \pi_1(\mathcal{DP}_1)\rangle\!\rangle$-chain exists. Moreover, $\pi_1(\mathcal{DP}_1)$ is ground, and hence $SN^{\rightsquigarrow_{R_1}}$ holds.

Example 4.18 Consider the constructor EV-TRS $R_5 = \{\ a \to d(c(y))\ \} \cup R_3$. Let π_5 be an AF function with $\pi_5(c) = [\ ]$. Now we have $\pi_5(R_5) = \{\ a \to d(c),\ d(0) \to 0,\ d(s(x)) \to s^2(d(x))\ \}$ and $\pi_5(\mathcal{DP}_{R_5}) = \{\ \langle A, D(c)\rangle, \langle D(s(x)), D(x)\rangle\ \}$. The inequalities $a \gtrsim_{\pi_5} d(c)$, $d(0) \gtrsim_{\pi_5} 0$, $d(s(x)) \gtrsim_{\pi_5} d(x)$, $A \succ_{\pi_5} D(c)$, and $D(s(x)) \succ_{\pi_5} D(x)$ hold where based quasi-ordering $\gtrsim$ is a recursive path ordering with $a > d$ and $A > D$. Therefore, $GSN^{\rightsquigarrow_{R_5}}$ holds. It is clear that $SN^{\rightsquigarrow_{R_5}}$ does not hold.

The following corollary is a little weaker but easier to use than Theorem 4.16.

Corollary 4.19 *Let R be an EV-TRS, $\leadsto_R$ have TRAT property and π be a simple AF that eliminates all extra variables of R and $\mathcal{DP}_R$ and that satisfies $\pi(\mathcal{DP}_R) = \mathcal{DP}_{\pi(R)}$. Then, $SN^{\rightarrow_{\pi(R)}}$ implies $GSN^{\leadsto_R}$. Moreover, if $\pi(t)$ is ground for all $\langle s, t\rangle \in \mathcal{DP}_R$ then $SN^{\leadsto_R}$.*

In Example 4.17, we have $\pi_1(\mathcal{DP}_{R_1}) = \mathcal{DP}_{\pi_1(R_1)}$, and $\pi_1(R_1)$ is ground. Hence, Corollary 4.19 is usable to prove $GSN^{\leadsto_{R_1}}$ and $SN^{\leadsto_{R_1}}$.

According to Proposition 4.4 (ii) and Theorem 4.5, for an EV-TRS R, each of followings is possible:

- to prove $GSN^{\leadsto_R}$ where R is right-linear,
- to prove that $SN_t^{\leadsto_R}$ for all linear terms t, where R is right-linear,
- to prove $GSN^{\leadsto_R}$ where R is a constructor system, and
- to prove $SN^{\leadsto_R}$ where R is a constructor system.

5 Related Works

In studies on normalizing reduction strategies [5,6,13,18], several kinds of EV-TRSs as approximations of TRSs are used. *Arbitral reduction systems* [5,9] can be formalized as EV-TRSs whose right-hand sides are extra variables. They introduced an Ω-reduction system to simulate the reduction sequence, which is a special case of narrowing extended in this paper. Although they are terminating, the theorems in Section 4 does not work to show their termination. The reason is that argument filtering method in this paper cannot eliminate all extra variables of collapsing rules. To overcome this problem is one of future works.

There are some studies on narrowing of conditional TRSs (CTRSs) with extra variables [4,11,12]. The targets of their results are 3-CTRSs, in which every extra variable must appear in condition parts. On the other hand, EV-TRSs belong to 4-CTRSs which are CTRSs with no restrictions, but not to 3-CTRSs. In addition, the CTRS, from which our motivating EV-TRS R_2 in Fig. 2 is obtained by transformation, is not 3-CTRS but 4-CTRS.

Some termination criteria for narrowing and E-narrowing have shown in [3]. The results in [3] treats TRSs in which the height of the left-hand side of rewrite rules is less than two.

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A EV-safe Reduction Sequences

Here, we give a precise definition of the EV-safe reduction sequences on EV-TRSs. Let t be a term and x be a variable. The set of all positions of x in t is denoted by $\mathcal{O}_x(t)$. Let $P, Q \subseteq \mathcal{O}(t)$. We write $P \leq Q$ if for all $q \in Q$ there exists some $p \in P$ such that $p \leq q$. The set $P \setminus p$ is defined as $P \setminus p = \{\, q \mid pq \in P \,\}$. The minimal set of P is defined as $\mathtt{min}(P) = \{\, p \mid p \in P, \neg(\exists q \in P, q < p) \,\}$. For example, $\mathtt{min}(\{11, 1, 2\}) = \{1, 2\}$. P is said to be *minimal* if $P = \mathtt{min}(P)$. We define the minimal set of union of P and Q as $P \sqcup Q = \mathtt{min}(P \cup Q)$, and the minimal set of intersection of P and Q as

$$P \sqcap Q = \{\, p \mid p \in \mathtt{min}(P), (\ \exists q \in \mathtt{min}(Q), q \leq p\ ) \,\} \cup \{\, q \mid q \in \mathtt{min}(Q), (\ \exists p \in \mathtt{min}(P), p \leq q\ ) \,\}.$$

For example, $\{11, 22\} \sqcup \{112, 2, 31\} = \{11, 2, 31\}$ and $\{11, 22\} \sqcap \{112, 2, 31\} = \{112, 22\}$.

We give the notion of the transition of positions at one-step reduction, adding the positions of extra variables.

Definition A.1 Let a rewrite rule $\rho : l \to r$, let P be a minimal set of positions and p be a position. We write $P \Rightarrow^{[p,\rho]} Q$ if there is no position q in P such that $q \leq p$, and $Q = \mathtt{min}(Q')$ for some Q' defined as follows:

$$Q' = (P \setminus \{\, q \mid p \leq q \,\}) \sqcup \left(\bigsqcup_{x \in \mathcal{EV}ar(\rho)} \{\, pq \mid q \in \mathcal{O}_x(r) \,\} \right) \sqcup \left(\bigsqcup_{x \in \mathcal{V}ar(l)} \{\, pqw \mid q \in \mathcal{O}_x(r), w \in (\textstyle\bigsqcap_{q' \in \mathcal{O}_x(l)} P \setminus pq') \,\} \right).$$

Note that if $P \subseteq \mathcal{O}(C[l\sigma]_p)$ then $Q \subseteq \mathcal{O}(C[r\sigma]_p)$.

This notion of transition is similar to that of descendants that follow redex positions [6]. We use a set of positions, such as P and Q, to represent positions, under which reductions are prohibited. The notation of $P \Rightarrow^{[p,\rho]} Q$ shows the transition in the one-step reduction at the position p by the rule ρ. For example, considering $P = \{11, 2\} \subseteq \mathcal{O}(h(f(s(y), 0, 0))$ and a rule $\rho : f(x, 0) \to g(x, x)$, we have $P \Rightarrow^{[1,\rho]} \{11, 12, 2\}$. Now, we define EV-safety as follows.

Definition A.2 Let R be an EV-TRS, $\rho_i : l_i \to r_i \in R$ and $P_0 \subseteq \mathcal{O}(s_0)$ be minimal. We say that the reduction sequence $s_0 \to_R^{[p_0,\rho_0]} s_1 \to_R^{[p_1,\rho_1]} \cdots$ is *EV-safe* with respect to P_0, if $P_0 \Rightarrow^{[p_0,\rho_0]} P_1 \Rightarrow^{[p_1,\rho_1]} \cdots$ holds. In the above case, we write $P_0 : s_0 \to_R^{[p_0,\rho_0]} P_1 : s_1 \to_R^{[p_1,\rho_1]} \cdots$. Especially, it is simply called *EV-safe* if $P_0 = \emptyset$.

Example A.3 Consider R_1 in Example 2.1. The sequence $g(0) \to_{R_1} h(0, 0) \to_{R_1} f(0, 0) \to_{R_1} s(0)$ is EV-safe because of

$$\emptyset : g(0) \to_{R_1} \{2\} : h(0, 0) \to_{R_1} \emptyset : f(0, 0) \to_{R_1} \emptyset : s(0).$$

On the other hand, the sequence $g(0) \to_{R_1} h(0, a) \xrightarrow{*}_{R_1} h(0, b)$ is not EV-safe since the subterm a of $h(0, a)$ is reduced.

B Proofs of Theorem 3.4 and 3.5

We prepare the following lemmas to prove the completeness. The first one can be easily proved.

Lemma B.1 *Let (σ, σ') be the most general unifier of terms s and t. For any unifier (θ, θ') of s and t, there exists a substitution δ such that $s\sigma\delta \equiv s\theta$ and $t\sigma'\delta \equiv t\theta'$.*

Lemma B.2 *Let R be an EV-TRS. Let $s\theta \to_R^{[p,\rho]} t$. If $s|_p$ is not a variable then there are a term t' and a substitution θ' such that $s \rightsquigarrow_R^{[p,\rho]} t'$ and $t \equiv t'\theta'$. Moreover, for a minimal set P such that $P \leq \mathcal{O}_{\mathcal{X}}(s)$, $P \Rightarrow^{[p,\rho]} Q$ implies $Q \leq \mathcal{O}_{\mathcal{X}}(t')$.*

Proof. Suppose that $\rho : l \to r \in R$, $s\theta \equiv C[l\sigma]_p$ and $t \equiv C[r\sigma]_p$. We can assume that $\mathcal{D}om(\theta) \cap \mathcal{D}om(\sigma) = \emptyset$ and $\mathcal{D}om(\theta) = \mathcal{V}ar(s)$ without loss of generality.

Let a context C' and a non-variable term v such that $s \equiv C'[v]_p$. Then we have $s\theta \equiv C'\theta[v\theta]_p \equiv C[l\sigma]_p$. It follows from $v\theta \equiv l\sigma$ that (θ, σ) is a unifier of s and $C[l]_p$. Then there exists the most general unifier (δ, σ'), and we have $\mathcal{VR}an(\delta) \cap (\mathcal{V}ar(s) \setminus \mathcal{D}om(\delta)) = \emptyset$ and $x\sigma' \notin \mathcal{V}ar(C\delta)$ for all $x \in \mathcal{EV}ar(\rho)$ from the definition of $\rightsquigarrow_R$. Hence we have $s \equiv C'[v]_p \rightsquigarrow_R C'\delta[r\sigma']_p$. On the other hand, it follows from $\mathcal{V}ar(C') \subseteq \mathcal{V}ar(s) = \mathcal{D}om(\theta)$ and Lemma B.1 that $v\delta\theta' \equiv l\sigma'\theta'$ for some substitution θ', and hence we have $r\sigma'\theta' \equiv r\sigma$ and $C'\delta\theta' \equiv C'\theta \equiv C$. Hence, $t \equiv C[r\sigma]_p \equiv C'\delta\theta'[r\sigma'\theta']_p \equiv (C'\delta[r\sigma']_p)\theta'$, which conclude the first part of the proof by taking $t' \equiv C'\delta[r\sigma]_p$.

Now, suppose $P \Rightarrow^{[p,\rho]} Q$, and we show that $Q \leq \mathcal{O}_{\mathcal{X}}(t')$. Let $q \in \mathcal{O}_{\mathcal{X}}(t')$. Consider the case that $p \not\leq q$. Since $q \not\leq p$ from $t' \equiv C'\delta[r\sigma']_p$, we have $q \in \mathcal{O}_{\mathcal{X}}(C'\delta)$. There exists $q' \leq q$ such that $q' \in \mathcal{O}_{\mathcal{X}}(C') \subseteq \mathcal{O}_{\mathcal{X}}(s)$. It follows from $p \in \mathcal{O}_{\mathcal{X}}(s)$ that $p' \leq q'$ for some $p' \in P$. Thus, $p' \in Q$ follows from $P \Rightarrow^{[p,\rho]} Q$. We have shown $p' \leq q$ and $p' \in Q$. Consider the case that $p \leq q$.

If q was introduced by means of an extra variable, that is $q = pq'$ and $r|_{q'} \equiv x \in \mathcal{EV}ar(\rho)$ for some q', we have $pq' \in Q$. Otherwise, q was moved via the reduction, that is $q = pq'w$ and $r|_{q'} \equiv y$ for some $y \in \mathcal{V}ar(l)$ and $w \in \mathcal{O}_{\mathcal{X}}(y\sigma')$. Then, we can show $p' \leq pq'w$ for some $p' \in Q$, from the fact that there exists $p'' \in P$ satisfying $p'' \leq pq'w$ for all q' such that $l|_{q'} \equiv y$. □

Theorem 3.4 *Let R be an EV-TRS. Let s and t be ground terms. If $s \xrightarrow{*}_R t$ is EV-safe then there exist a term t' and substitution θ such that $s \stackrel{*}{\rightsquigarrow}_R t'$ and $t \equiv t'\theta$.*

Proof. From Lemma B.2, we can easily prove the following claim by induction on n: if $P : s'\theta \xrightarrow{n}_R P' : t$ and $P \leq \mathcal{O}_{\mathcal{X}}(s)$, then $s' \stackrel{*}{\rightsquigarrow}_R t'$ and $t'\theta' \equiv t$ for some t' and θ'. □

Theorem 3.5 *Let R be a right-linear EV-TRS. Let s and t be ground terms. Then, $s \xrightarrow{*}_R t$ implies $s \stackrel{*}{\rightsquigarrow}_R t'$ and $t \equiv t'\theta$ for some linear term t' and substitution θ.*

Proof. We prove by induction on n that $s \xrightarrow{n}_R t$ implies $s \stackrel{*}{\rightsquigarrow}_R t'$ and $t \equiv t'\theta$ for some linear term t' and some substitution θ. The case of $n = 0$ is trivial. Suppose $s \xrightarrow{n-1}_R u \rightarrow_R t$. By induction hypothesis, there exist a linear term u' and a substitution θ' such that $s \stackrel{*}{\rightsquigarrow}_R u'$ and $u'\theta' \equiv u$.

- Consider the case $p \in \mathcal{O}_{\mathcal{F}}(u')$. Then, we have $u'\theta' \equiv C[l\sigma]_p$. From the first part of Lemma B.2 and from Proposition 4.4, there exist a linear term t' and a substitution θ' such that $u' \rightsquigarrow_R t'$ and $t \equiv t'\theta'$.
- Consider the case $p \in \mathcal{O}_{\mathcal{X}}(u')$. Suppose $u \equiv C[l\sigma]_p \rightarrow_R^{[p,\rho]} C[r\sigma] \equiv t$ where $\rho : l \rightarrow r \in R$. Then, we have $u'|_q \equiv y$ and $p = qq'$ for some $y \in \mathcal{V}ar(u')$, q and q'. From the linearity of u', we have $u' \equiv C'[y]_q$ and $y\theta' \equiv C''[l\sigma]_{q'}$ for some C' and C''. Let $\theta = \theta'|_{\mathcal{D}om(\theta')\setminus\{y\}} \cup \{y \mapsto C''[r\sigma]_{q'}\}$. Then, θ is a substitution, and we have $s \stackrel{*}{\rightsquigarrow}_R u'$ and $u'\theta \equiv C'\theta[y\theta]_q \equiv C[r\sigma]_p \equiv t$.

□

C Proofs of Theorem 4.5, 4.9 and Lemma 4.14

Let $s \rightsquigarrow_R^q t$. Then, we write $s \rightsquigarrow_R^{p<} t$ if $p < q$, and write $s \rightsquigarrow_R^{p\leq} t$ if $p \leq q$. A substitution θ is said to be $SN^{\rightarrow}$, written as $SN^{\rightarrow}_{\theta}$, if $SN^{\rightarrow}_{x\theta}$ for all $x \in \mathcal{D}om(\theta)$.

Theorem 4.5 *Let R be a constructor EV-TRS. Then, $\rightsquigarrow_R$ has TRAT property.*

Proof. For a constructor system R, the followings hold obviously.

(i) Let $SN_t^{\rightsquigarrow_R}$ and $t \;{}_{\delta}\!\!\rightsquigarrow_R t'$. Then, $SN_{\delta}^{\rightsquigarrow_R}$.

(ii) $SN_t^{\rightsquigarrow_R}$ implies $SN_{t\theta}^{\rightsquigarrow_R}$ for all substitution θ with $SN_{\theta}^{\rightsquigarrow_R}$.

Assuming that there exists an almost terminating sequence $t \equiv t_0 \rightsquigarrow_R^{\varepsilon<} t_1$

$\rightsquigarrow_R^{\varepsilon<} \cdots$ which is not top reduced, we show a contradiction that there exists a proper subterm u of t that $\neg SN_u^{\rightsquigarrow_R}$. Let $t_i \equiv f(t_{i,1}, \ldots, t_{i,n})$ without loss of generality. From the infinite sequence, for every i, there exists j such that $t_{i,j} \ {}_{\delta_i}\!\!\rightsquigarrow_R t_{i+1,j}$, $SN_{\delta_i}^{\rightsquigarrow_R}$, and for every j' with $j' \neq j$, $t_{i,j'}\delta_i \equiv t_{i+1,j'}$. Then, we have infinite number of narrowing derivations below k for at least one position k with $1 \leq k \leq n$. Every step from $t_{i,k}$ to $t_{i+1,k}$ consists of either $t_{i,k} \ {}_{\delta_i}\!\!\rightsquigarrow_R t_{i+1,k}$ or $t_{i,k}\delta_i \equiv t_{i+1,}$. Since narrowing derivations below k is infinite, it follows from the contraposition of (ii) that $\neg SN_{t_{i,k}}^{\rightsquigarrow_R}$, and hence $\neg SN_{t_{0,k}}^{\rightsquigarrow_R}$. Therefore, there is a proper subterm $t_{0,k}$ of t that $\neg SN_{t_{0,k}}^{\rightsquigarrow_R}$. □

We abbreviate the sequence $a_{i,1}, \ldots, a_{i,n_i}$ as $\vec{a_i}$.

Theorem 4.9 *Let R be an EV-TRS and $\rightsquigarrow_R$ has TRAT property.*

(i) *$SN^{\rightsquigarrow_R}$ if and only if there is no infinite $\langle\!\langle \rightsquigarrow_R, \mathcal{DP}_R \rangle\!\rangle$-chain.*

(ii) *$GSN^{\rightsquigarrow_R}$ if and only if there is no infinite ground $\langle\!\langle \rightsquigarrow_R, \mathcal{DP}_R \rangle\!\rangle$-chain.*

Proof. We prove only the claim (ii) since the proof of (i) is similar to (ii).

To show the *if*-part of (ii), we construct an infinite ground $\langle\!\langle \rightsquigarrow_R, \mathcal{DP}_R \rangle\!\rangle$-chain from an infinite ground narrowing sequence. Since $\rightsquigarrow_R$ has TRAT property, we can assume the infinite sequence is a top reduced almost terminating narrowing sequence and starts from ground term $s_0 \equiv f_1(\vec{u_0})$. Then, we have

$$f_1(\vec{u_0}) \overset{*}{\rightsquigarrow}_R^{\varepsilon<} f_1(\vec{v_1}) \equiv s'_1 \ {}_{\delta_1}\!\!\rightsquigarrow_R^{[\varepsilon,\rho_1]} r_1\sigma_1 \rightsquigarrow_R \cdots$$

where $\rho_1 : f_1(\vec{w_1})(\equiv l_1) \to r_1 \in R$ and $(\delta_1, \sigma_1) = \mathtt{mgu}(s'_1, l_1)$. Since $SN_{v_{1,i}}^{\rightsquigarrow_R}$ holds, $SN_{x\sigma_1}^{\rightsquigarrow_R}$ holds for any $x \in \mathcal{D}om(\sigma_1)$. Hence, there is a subterm $t_1 \equiv f_2(\vec{u_1})$ of r_1 such that there exists a top reduced almost terminating sequence starting from $t_1\sigma_1$ since $\rightsquigarrow_R$ has TRAT property. Then, as similar as the case of s_0, we have

$$t_1\sigma_1 \equiv f_2(\vec{u_1}\sigma_1) \overset{*}{\rightsquigarrow}_R^{\varepsilon<} f_2(\vec{v_2}) \equiv s'_2 \ {}_{\delta_2}\!\!\rightsquigarrow_R^{[\varepsilon,\rho_2]} r_2\sigma_2 \rightsquigarrow_R \cdots$$

where $\rho_2 : f_2(\vec{w_2})(\equiv l_2) \to r_2 \in R$ and $(\delta_2, \sigma_2) = \mathtt{mgu}(s'_2, l_2)$. Since $SN_{v_{2,i}}^{\rightsquigarrow_R}$ holds, $SN_{x\sigma_2}^{\rightsquigarrow_R}$ also holds for any $x \in \mathcal{D}om(\sigma_2)$. Hence, there is a subterm $t_2 \equiv f_3(\vec{u_2})$ of r_2 such that there exists a top reduced almost terminating sequence starting from $t_2\sigma_2$. Here, $\langle F_1(\vec{w_1}), F_2(\vec{u_2})\rangle$, $\langle F_2(\vec{w_2}), F_3(\vec{u_3})\rangle \in \mathcal{DP}_R$ follow from ρ_1 and ρ_2. Since $u_{0,i} \overset{*}{\rightsquigarrow}_R v_{1,i}$, $(\delta_1, \sigma_1) = \mathtt{mgu}(s'_1, l_1)$, $u_{1,i} \overset{*}{\rightsquigarrow}_R v_{2,i}$ and $(\delta_2, \sigma_2) = \mathtt{mgu}(s'_2, l_2)$, we have a ground chain $F_1(\vec{u_0})\langle F_1(\vec{w_1}), F_2(\vec{u_2})\rangle\langle F_2(\vec{w_2}), F_3(\vec{u_3})\rangle$.

By repeating the above argument, we obtain an infinite ground chain

$$F_1(\vec{u_0})\langle F_1(\vec{w_1}), F_2(\vec{u_2})\rangle\langle F_2(\vec{w_2}), F_3(\vec{u_3})\rangle \cdots.$$

We prove *only-if*-part of (ii) by constructing an infinite ground narrowing-sequence from an infinite ground $\langle\!\langle \rightsquigarrow_R, \mathcal{DP}_R \rangle\!\rangle$-chain

$$F_1(\vec{u_0})\langle F_1(\vec{w_1}), F_2(\vec{u_2})\rangle\langle F_2(\vec{w_2}), F_3(\vec{u_3})\rangle \cdots.$$

From the definition of $\langle\!\langle \rightsquigarrow_R, \mathcal{DP}_R \rangle\!\rangle$-chain, there are a term $F_i(\vec{v_i})$ and the most general unifier $(\delta_i, \sigma_i) = \mathtt{mgu}(F_i(\vec{v_i}), F_i(\vec{w_i}))$ such that $F_i(\vec{u_i})\sigma_{i-1} \overset{*}{\rightsquigarrow}_R F_i(\vec{v_i})$, where $F_1(\vec{u_0})\sigma_0 \equiv F_1(\vec{u_0})$. From the construction of dependency pairs, we have

$\rho_i : f_i(\vec{w_i}) \to C_i[f_{i+1}(\vec{u_{i+1}})] \in R$. Hence, we can easily construct an infinite ground narrowing-sequence

$$f_1(\vec{u_0}) \overset{*}{\leadsto}_R f_1(\vec{v_1}) \leadsto_R^{[\varepsilon,\rho_1]} C_1\delta_1[f_2(\vec{u_1})]_{p_1} \overset{*}{\leadsto}_R^{p_1<} C_1\delta_1[f_2(\vec{v_2})]_{p_1}$$
$$\leadsto_R^{[p_1,\rho_2]} C_1\delta_1\delta_2[C_2\delta_2[f_3(\vec{u_2})\sigma_2]_{p_2}]_{p_1} \leadsto_R \cdots.$$

□

Let π be a simple AF and θ be a substitution. We define the substitution θ_π as $\theta_\pi = \{\, x \mapsto \pi(x\sigma) \mid x \in \mathcal{D}om(\theta) \,\}$. Let t be a term. It is clear that $\pi(t\theta) \equiv \pi(t)\theta_\pi$.

Lemma 4.14 *Let R be an EV-TRS, π be a simple AF function that eliminates all extra variables of R and $\mathcal{DP}_R$. Let s and t be terms such that $\pi(s)$ is ground. Then, $s \overset{*}{\leadsto}_R t$ implies $\pi(s) \overset{*}{\to}_{\pi(R)} \pi(t)$.*

Proof. We prove by induction on n of $s \overset{n}{\leadsto}_R t$.

Since the case of $n = 0$ is trivial, we assume that $s\ {}_{\delta}\!\leadsto_R^{[p,\rho]} u\ {}_{\delta'}\!\overset{n-1}{\leadsto}_R t$ and $\pi(s)$ is ground, where $\rho : l \to r \in R$. Then, there are a context C, a term s' and the most general unifier $(\delta, \sigma) = \mathtt{mgu}(s, C[l]_p)$ such that $s \equiv C[s']_p$ and $u \equiv C\delta[r\sigma]_p$. Since $\pi(s)$ is ground, $\pi(C)$ is also ground.

- Consider the case that $\Box$ in C is eliminated by π. Now we have $\pi(s) \equiv \pi(C[s']_p) \equiv \pi(C)$ and $\pi(u) \equiv \pi(C\delta[r\sigma]_p) \equiv \pi(C\delta) \equiv \pi(C)\delta_\pi \equiv \pi(C)$. By induction hypothesis, we have $\pi(u) \overset{*}{\to}_{\pi(R)} \pi(t)$. Therefore, $\pi(s) \equiv \pi(u) \overset{*}{\to}_{\pi(R)} \pi(t)$.
- Otherwise. Since $\pi(C)$ is ground and $\pi(C)$ is a context, we have $\pi(s) \equiv \pi(C[s']_p) \equiv (\pi(C)[\pi(s')]_q$ and $\pi(u) \equiv \pi(C\delta[r\sigma]_p) \equiv (\pi(C\delta))[\pi(r\sigma)]_q \equiv \pi(C)\delta_\pi[\pi(r)\sigma_\pi]_q \equiv \pi(C)[\pi(r)\sigma_\pi]_q$. On the other hand, $\pi(s')\delta_\pi \equiv \pi(l)\sigma_\pi$ follows from $s'\delta \equiv l\sigma$, $\pi(s'\delta) \equiv \pi(s')\delta_\pi$ and $\pi(l\sigma) \equiv \pi(l)\sigma_\pi$. Since $\pi(s')$ is ground, we have $\pi(s')\delta_\pi \equiv \pi(s') \equiv \pi(l)\sigma_\pi$. We also have $\pi(l) \to \pi(r) \in \pi(R)$. It follows from the assumption that $\mathcal{V}ar(\pi(l)) \supseteq \mathcal{V}ar(\pi(r))$ for every $l \to r \in R$, and hence $\pi(R)$ is a TRS. Then, $\pi(s') \equiv \pi(l)\sigma_\pi \to_{\pi(R)} \pi(r)\sigma_\pi$ and $\pi(r)\sigma_\pi$ is ground. Since $\pi(u) \equiv \pi(C)[\pi(r)\sigma_\pi]$ is also ground, we have $\pi(u) \overset{*}{\to}_{\pi(R)} \pi(t)$ by induction hypothesis. Therefore, we have the sequence $\pi(s) \equiv \pi(C)[\pi(l)\sigma_\pi]_q \to_{\pi(R)} \pi(C)[\pi(r)\sigma_\pi]_q \equiv \pi(u) \overset{*}{\to}_{\pi(R)} \pi(t)$.

□