計算理工学専攻修士学位論文

Study on Inversion of Term Rewriting Systems

— 項書換え系の逆計算に関する研究 —

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Abstract

There are many cases that we need a (conditional) term rewriting system (written as (C)TRS, for short) which computes the inverses of the functions defined by another (C)TRS. Such a CTRS is called an inverse system. Inverse systems are useful in such a case that a CTRS cannot compute the proper values of given terms while it seems natural and correct. That is, inverse functions may be used to transform the CTRS to an equivalent CTRS which can compute the proper values of all terms. For example, consider the rule $f(x + y, y) \to f(x, y)$, which cannot apply to any integer because the first argument of f in the left-hand side contains $+$ and no integer contains $+$. This rule can be transformed equivalently to $f(z, y) \to f(z-y, y)$ by replacing the first argument $x+y$ of f in the left-hand side with a fresh variable z and the first argument x of f in the right-hand side with $z-y$, where $-$ is the inverse of $+_y$ defined as $+_y(x) = x+y$. In this example, we fortunately know the CTRS which computes $-$ i.e., the inverse of $+_y$, and can do the transformation mechanically. However, we cannot assume that all definitions of necessary inverse functions are given. Therefore, it is desirable to automatically construct a CTRS of inverse functions from a given CTRS.

This paper proposes a method that constructs an inverse system of a given constructor TRS. The inverse systems that our method constructs are generally CTRSs. We present a transformation of our inverse CTRSs into equivalent TRSs. We also show a result on termination of the inverse systems obtained by our method. We finally show an application of inverse systems to CTRS transformation.

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Chapter 1

Introduction

There are many cases that we need a (conditional) term rewriting system (written as (C)TRS, for short) which computes the inverses of the functions defined by another (C)TRS. Consider the following CTRS:

$$R_{\mathsf{gcd}} = \{ \begin{array}{lll} gcd(x+y,y) \to gcd(x,y), & & \\ gcd(x,0) \to x, & gcd(x,y) \to gcd(y,x) \Leftarrow x < y, & \\ 0 + y \to y, & s(x)+y \to s(x+y) & \} . \end{array}$$

The function gcd is intended to compute the greatest common divisor of two natural numbers. However, the term $gcd(s^6(0), s^2(0))$ is not reduced to $s^2(0)$ since there is no rule which matches $gcd(s^6(0), s^2(0))$. Thus R_{gcd} cannot compute gcd although it seems natural and correct when regarded as an equational specification.

If there is a CTRS that computes the subtraction $-$, we can transform R_{gcd} to the following CTRS R'_{gcd} by replacing the first argument $x+y$ of the left-hand side in the first rule of R_{gcd} with a variable z and the first argument x of the right-hand side in it with $z-y$:

$$R'_{\mathsf{gcd}} = \{ \begin{array}{lll} gcd(z,y) \to gcd(z-y,y), & & \\ gcd(x,0) \to x, & gcd(x,y) \to gcd(y,x) \Leftarrow x < y, & \\ y - y \to 0, & s(x)-y \to s(x-y) & \} . \end{array}$$

Here, $-$ is the inverse function of $add_y(x) = x + y$ in the sense that $z - y = add_y^{-1}(z)$. R'_{gcd} computes $gcd(s^6(0), s^2(0))$; $gcd(s^6(0), s^2(0)) \xrightarrow{*}_{R'_{\mathsf{gcd}}} s^2(0)$. Fortunately, we have known that the inverse of add_y is $-$ in this case. However, we cannot assume that all definitions of necessary inverse functions are given. Therefore, it is desirable to automatically construct such as R'_{gcd} from R_{gcd}. The key issue is how to construct a CTRS for the inverse function of a partially-fixed argument function, like add_y, that is defined by another CTRS, e.g., how to construct a CTRS for $-$, the ivnerse of $add_y(x)$, from a CTRS for $+$.

This paper proposes a method which constructs an inverse system of a given constructor TRS. Here we mean, by an inverse system of a CTRS R, a CTRS that computes the inverses of partially-fixed argument functions of the functions computed by R. The inverse systems that our method constructs are generally CTRSs. We introduce a transformation of generated CTRSs into TRSs. We also give a result on termination of generated inverse systems. We show an application of inverse systems to CTRS transformation as described at the beginning of this chapter.

This paper is organaized as follows. In Chapter 3, we introduce our basic idea by using natural inverse CTRSs of pure treeless TRSs, present a method which generates a natural inverse CTRSs, that is an inverse system regarding multiple arguments as tupled single argument, and prove correctness of the method if the input are ground-convergent constructor 3-TRSs. In Chapter 4, we extend the method of natural inverse CTRSs to that of indexed inverse systems, that is an inverse system of partially-fixed argument functions of each function of the input TRS, then prove correctness of extended method for input ground-convergent constructor 3-TRSs. Chapter 5 introduces the transformation of inverse CTRSs into inverse TRSs. Chapter 6 discusses termination of generated inverse systems. In Chapter 7, we show an application of our algorithms to CTRS transformation and to solve equations by using inverse systems of deteministic 3-CTRSs. Inverse CTRSs of pure treeless TRSs are the restricted result of this paper found in [NSS01a, NSS01b, NSS01d, NSS02], and inverse systems proposed in this paper are the extended result of [NSS01c].

Chapter 2

Preparation

We mainly follow the notation of [BN98, Kl92]. A signature is a set F of function symbols accompanied with a mapping *arity* from F to the set of natural numbers. Let X be a set of variables. Terms over F and X are recursively defined as follows: a variable $x \in X$ is a term, and if $f \in F$, $arity(f) = n$ and $t_1, ..., t_n$ are terms then $f(t_1, ..., t_n)$ is a term. Note that if $arity(f) = 0$ then $f()$ is a term, which is called a *constant term* and written as f. We often use s, t, u to stand for a term. $T(F, X)$ denotes the set of all terms over F and X. $\mathcal{V}ar(t)$ denotes the set of all variables which appear in t. If $\mathcal{V}ar(t) = \emptyset$ then t is said to be *ground*. A set $T(F, \emptyset)$ of ground terms is abbreviated to $T(F)$. A term is called *linear* if every variable occurs in it at most once. The identity of terms s, t is denoted by $s \equiv t$. If $t \equiv f(t_1, ..., t_n)$ then $top(t)$ denotes the top symbol f. We use $f^n(t)$ to denote $\underbrace{f(\cdots(f(}_{n}t))\cdots)$.

Let $\Box$ be an extra function symbol not included in F and let $arity(\Box) = 0$. A *context* is a term $C \in T(F \cup \{\Box\}, X)$, which is denoted by $C[, ...,]$. If $C[, ...,]$ has exactly one $\Box$ then it is written as $C[\ ]$. If $C[, ...,]$ contains n-$\Box$'s then $C[t_1, ..., t_n]$ denotes the term obtained by replacing $\Box$'s with terms $t_1, ..., t_n$ from left to right. If $t \equiv C[s]$, we say that s is a *subterm* of t and write as $s \trianglelefteq t$. We also write $f \dot{\in} t$ if there exists a term s such that $s \trianglelefteq t$ and $top(s) = f$.

A *substitution* is a mapping $\sigma : X \to T(F, X)$ such that $\mathcal{D}om(\sigma)$ is finite, where $\mathcal{D}om(\sigma) = \{x \mid \sigma(x) \not\equiv x\}$. If $\mathcal{D}om(\sigma) = \{x_1, ..., x_n\}$ and $\sigma(x_i) \equiv u_i$ for $i = 1, ..., n$ then σ is denoted by $\{x_1 \mapsto u_1, ..., x_n \mapsto u_n\}$. For any term t, $t\sigma$

denotes the term obtained by simultaneously replacing x_i in t with u_i for all i.

An *(oriented) conditional rewrite rule* is a triple $(l, r, \mathcal{C}ond)$, denoted by $l \to r \Leftarrow \mathcal{C}ond$, where l $(\notin X)$ and r are terms called *left-hand side* and *right-hand side*, respectively, and $\mathcal{C}ond$ is a formula of the form $l_1 \to r_1 \wedge \cdots \wedge l_n \to r_n$. If $n = 0$ then $l_1 \to r_1 \wedge \cdots \wedge l_n \to r_n$ is denoted by $\texttt{true}$. A relation $\to$ over terms and a substitution σ satisfy $\mathcal{C}ond$, written as $\mathcal{C}ond(\sigma, \to)$, if $l_i\sigma \xrightarrow{*} r_i\sigma$ for all i, where $\xrightarrow{*}$ is the reflexive and transitive closure of $\to$. An *(oriented) conditional term rewriting system* (*CTRS*) is a set of conditional rewrite rules. We say that a CTRS R is *over* $T(F,X)$ if $top(l) \in F_D$, $l, r \in T(F,X)$ and terms appearing in $Cond$ are in $T(F,X)$ for every rule $l \to r \Leftarrow Cond \in R$. For a CTRS R, we define $\underset{n}{\to}$ by induction as follows: (1) $\underset{0}{\to} = \emptyset$, (2) $\underset{n+1}{\to} = \underset{n}{\to} \cup \{\, (C[l\sigma], C[r\sigma]) \mid C$ is a context, σ is a substitution, $l \to r \Leftarrow \mathcal{C}ond \in R, \mathcal{C}ond(\sigma, \underset{n}{\to}) \,\}$. We call $\underset{n}{\to}$ the level n reduction of R. Define $\to_R = \bigcup_{n=0}^{\infty} \underset{n}{\to}$. If there is no s such that $t \to_R s$ then t is called a *normal form* (*n.f.*, for short) with respect to R. We write a set of normal forms with respcet to R as $NF_R(F,X)$, and abbreviate $NF_R(F,\emptyset)$ to $NF_R(F)$. A term $l\sigma$ is called a *redex* if $Cond(\sigma, \to)$ for a rule $l \to r \Leftarrow Cond \in R$.

For a conditional rewrite rule $l \to r \Leftarrow Cond$, variables in $(\mathcal{V}ar(r) \cup \mathcal{V}ar(Cond)) - \mathcal{V}ar(l)$ are called *extra variables*. $\mathcal{EV}ar_{l \to r \Leftarrow Cond}(t)$ denotes the set of extra variables in t which is a subterm appearing in l, r or $Cond$. We abbreviate it to $\mathcal{EV}ar(t)$ if that causes no confusion. A *1-CTRS* has no extra variables in each rule, a *2-CTRS* has no extra variables on the right-hand side of each rule, a *3-CTRS* may contain extra variables on the right-hand sides of the rules provided that these also occur in the corresponding conditional part (i.e., $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(c)$), and a *4-CTRS* may contain extra variables without any restriction.

A conditional rewrite rule $l \to r \Leftarrow Cond$ is *left-linear* (and *right-linear*, resp.) if any variable occurs in the left-hand side l (and the right-hand side r, resp.) at most once. A CTRS is *left-linear* (and *right-linear*, resp.) if every rule is left-linear (and right-linear, resp.). The rules $l \to r \Leftarrow \mathcal{C}ond$ and $l' \to r' \Leftarrow \mathcal{C}ond'$ *overlap* if there exist a subterm s (not a variable) of l' and substitutions σ and σ' such that $l\sigma \equiv s\sigma'$. A CTRS is *orthogonal* if it is left-linear and has no overlapping rules. We abbreviate a rule $l \to r \Leftarrow \texttt{true}$ as $l \to r$, and call it a *rewrite rule*. A *term rewriting system* (*TRS*) is a CTRS that has only rewrite rules. We have $\rightarrow_R = \underset{i}{\rightarrow}$ for all $i \geq 1$ on TRS R. Let a CTRS R over $T(F, X)$. A term $s \in T(F, X)$ is *confluent* on R if $s \xrightarrow{*}_R t$ and $s \xrightarrow{*}_R t'$ imply some term $u \in T(F, X)$ such that $t \xrightarrow{*}_R u$ and $t' \xrightarrow{*}_R u$ for all $t, t' \in T(F, X)$. R is called confluent if every term over $T(F, X)$ is confluent on R. An orthogonal TRS is known to be confluent. A CTRS R is called *terminating* if there exists no infinite sequecne $t_0 \rightarrow_R t_1 \rightarrow_R t_2 \cdots$. A terminating CTRS is said to be *ground-convergent* if all ground terms are confluent.

Chapter 3

Natural Inverse CTRSs

We first consider natural inverse functions like f^{-1} that computes values $v_1, ..., v_n$ from a value v for $f(v_1, ..., v_n) = v$. In this chapter, we introduce our basic idea by using natural inverse CTRSs of pure treeless TRSs, then give the definition of natural inverse CTRSs and present how to generate natural inverse CTRSs from constructor TRSs. Then, we prove the correctness of the method for natural inverse CTRSs of ground-convergent constructor 3-TRSs.

We suppose that the signature F is divided to disjoint sets F_D of *defined symbols* and F_C of *constructors*, that is $F = F_D \oplus F_C$. We call $t \in T(F_C, X)$ a *constructor term*. We use f, g, h as defined symbols, c, d, e as constructors, v, w, x, y, z as variables and p, q as constructor terms. Let F be a signature and X be a set of variables. We use $C[\![t_1, ..., t_n]\!]$ for representing $C[t_1, ..., t_n]$ with $C \in T(F_C \cup \{\Box\}, X)$, $C \not\equiv \Box$ and $top(t_i) \in F_D$. For a term t such that $top(t) \notin F_D$, we have $t \equiv C[\![t_1, ..., t_n]\!]$ with some $t_1, ..., t_n \in T(F, X)$ and some context C. We introduce special constructors tp_n in order to bundle the return values of the inverses of multiple argument functions ; a *tuple* of n terms $t_1, ..., t_n$ is denoted by $tp_n(t_1, ..., t_n)$. A tuple $tp_1(t)$ is sometimes simply denoted by t. Let R be a CTRS over $T(F, X)$. For $f \in F_D$ and $t_1, ..., t_n \in T(F_C)$, we say that $f(t_1, ..., t_n)$ is *defined* if $f(t_1, ..., t_n) \xrightarrow{*}_R s$ for some $s \in T(F_C)$.

An orthogonal left-linear TRS R over $T(F, X)$ is said to be a *pure treeless TRS* if every rule of f is represented as follows:

$$f(p, x_2, ..., x_n) \to C[\![f_1(x_{1,1}, ..., x_{1,n_1}), ..., f_m(x_{m,1}, ..., x_{m,n_m})]\!] \in R$$

and p is either a variable x_0 or a simple constructor term $c(x_{0,1}, ..., x_{0,n_0})$ for all $f \in F_D$.

A CTRS R over $T(F, X)$ is said to be a *constructor CTRS* if $f \in F_D$ and $t_1, ..., t_n \in T(F_C, X)$ for every rule $f(t_1, ..., t_n) \to r \Leftarrow \in R$. Note that constructor CTRSs are not always orthogonal. A pure treeless TRS is a restricted constructor TRS on expression. For exapmle, a pure treeless TRS computing multiplication is as follows:

$$
\begin{aligned}
R_{\mathsf{pt}} = \{ \quad & mult_{\mathsf{pt}}(0, y) \to 0, \\
& mult_{\mathsf{pt}}(s(x), y) \to addmult(y, x, y), \\
& addmult(0, y, z) \to mult_{\mathsf{pt}}(y, z), \\
& addmult(s(x), y, z) \to s(addmult(x, y, z)) \quad \}.
\end{aligned}
$$

This definition of multiplication is not natural but correct (compare R_1 of Example 1). This paper treats (mainly, ground-convergent) constructor 3-TRSs, simply called constructor TRSs, as inputs of our method. since the class defined by pure treeless TRSs are not clear.

Example 1 *A ground-convergent constructor 3-TRS R_1 implementing the function mult which computes multiplication is defined as follows:*

$$
\begin{aligned}
R_1 = \{ \quad & add(0, y) \to y, \quad add(s(x), y) \to s(add(x, y)), \\
& mult(0, y) \to 0, \quad mult(s(x), y) \to add(mult(x, y), y) \ \}. \qquad \square
\end{aligned}
$$

3.1 Definition of Natural Inverse CTRSs

Firstly, we define natural inverse CTRSs.

Definition 1 *Let R be a CTRS over $T(F, X)$, $F_0 \subseteq F_D$ and R' be a CTRS over $T(G, X')$ such that $G_C = F_C \cup \{tp_i \mid 1 \leq i \leq n\}$ and $G_D = F_D \oplus G'_D$, where n is the maximum number of arity of $f \in F_D$. R' is a natural inverse of R with respect to F_0 if for any $f \in F_0$ there exists $g \in G'_D$ such that $f(t_1, ..., t_n) \xrightarrow{*}_R s$ implies $g(s) \xrightarrow{*}_{R'} tp_n(t_1, ..., t_n)$ for all $t_1, ..., t_n, s \in T(F_C)$. In this case, we say that g is a natural inverse function of f.*

Example 2 *Consider the following two TRSs:*

$$R_2 = \{ \quad double(0) \to 0, \quad double(s(x)) \to s(s(double(x))) \quad \},$$

$$R_3 = \{ \quad half(0) \to 0, \quad half(s(s(x))) \to s(half(x)) \quad \}.$$

R_3 is a natural inverse of R_2 with respect to $\{double\}$. For example, we have $double(s^2(0)) \xrightarrow{*}_{R_2} s^4(0)$ and $half(s^4(0)) \xrightarrow{*}_{R_3} s^2(0)$. □

3.2 Basic Idea

We explain our basic idea by using a unary function. Let's consider the pure treeless TRS consisting of the rule $f(x) \to C[\![f_1(x), ..., f_m(x)]\!]$.

Firstly, applying the inverse function symbol f^{-1} of f to the both hand sides, the left-hand side is transformed to x since "$f^{-1}(f(n)) = n$" from the desired property of inverse functions. Then, we have $x \to f^{-1}(C[\![f_1(x), ..., f_m(x)]\!])$. Secondly, we exchange the both hand sides and replace the defined symbol $f_i(x)$ in the right-hand side with a fresh variable y_i for each i. Then, we have $x \to f^{-1}(C[y_1, ..., y_m])$. Finally, we need to add equations $x \equiv f_1^{-1}(y_1)$,..., $x \equiv f_m^{-1}(y_m)$ as the condition of the rule in order to relate x to y_i's. Thus, we obtain the conditional rewrite rule $f^{-1}(C[y_1, ..., y_m]) \to x \Leftarrow \bigwedge_{i=1}^{m} f_i^{-1}(y_i) \to x$ that defines the inverse function of f.

3.3 Generation of Natural Inverse CTRSs

Secondly, we give a method to generate natural inverse CTRSs from constructor TRSs, extending the basic idea in Sectin 3.2 for the rules $f(p_1, ..., p_n) \to C[\![t_1, ..., t_n]\!]$ of constructor TRSs. In the sequel, we use # to represent inverse function symbols; $f^{\#}$ is a natural inverse of f. The algorithm generating natural inverse CTRSs of pure treeless TRSs is presented in [NSS01a, NSS01b, NSS01d, NSS02].

Let R be a constructor TRS over $T(F, X)$ and $F_0 \subseteq F_D$. At the first step, we define the set $\mathcal{R}eq_{\mathsf{N}}(R, F_0)$ of the defined symbols required to compute defined symbols in F_0.

Definition 2 *Let R be a constructor TRS over $T(F,X)$ and $F_0 \subseteq F_D$. We define $\mathcal{R}eq_{\mathsf{N}}(R,F_0)$ to be the smallest set satisfying the followings:*

- $\mathcal{R}eq_{\mathsf{N}}(R,F_0) \supseteq F_0$,
- $\mathcal{R}eq_{\mathsf{N}}(R,F_0) \supseteq \{\, f \mid l \to r \in R, top(l) \in \mathcal{R}eq_{\mathsf{N}}(R,F_0), r \mathrel{\dot{\ni}} f \in F_D \,\}$.

Example 3 *For R_1 of Example 1, $\mathcal{R}eq_{\mathsf{N}}(R_1, \{\, mult \,\}) = \{\, add, mult \,\}$.* □

Next, we define the natural inverse CTRS $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R,F_0)$ of a constructor TRS R over $T(F,X)$ with respect to F_0 ($\subseteq F_D$). $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ is an extension of the algorithm that generates natural inverse CTRSs of pure treeless TRSs[NSS01d].

Definition 3 *Let R be a constructor TRS over $T(F,X)$ and $F_0 \subseteq F_D$. The signature G ($= G_D \oplus G_C$) is defined as follows:*

$$G_D = \{\, f^{\#}, f \mid f \in \mathcal{R}eq_{\mathsf{N}}(R,F_0) \,\},$$

$$G_C = F_C \cup \{\, tp_i \mid 0 \le i \le \max_{f \in F_D}(arity(f)) \,\}.$$

The natural inverse CTRS $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R,F_0)$ of R with respect to F_0 is defined over $T(G,X')$ as follows:

- $$\begin{aligned}\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R,F_0) &= \{\, \mathcal{I}nv\mathcal{R}ule_{\mathsf{N}}(\, l \to r \,) \mid l \to r \in R, top(l) \in \mathcal{R}eq_{\mathsf{N}}(R,F_0) \,\} \\ &\quad \cup \{\, f^{\#}(f(x_1,...,x_n)) \to tp_n(x_1,...,x_n) \mid f \in \mathcal{R}eq_{\mathsf{N}}(R,F_0) \,\},\end{aligned}$$
- $\mathcal{I}nv\mathcal{R}ule_{\mathsf{N}}(f(p_1,...,p_n) \to r) = f^{\#}(r') \to tp_n(p_1,...,p_n) \Leftarrow \mathcal{C}ond$,
  where $\mathcal{T}_{\mathsf{N}}(\!|\, r \,|\!) = \langle\, r' \,;\, \mathcal{C}ond \,\rangle$.

The procedure $\mathcal{T}_{\mathsf{N}}$ that takes a term t as input and outputs the pair $\langle\, t' \,;\, \mathcal{C}ond \,\rangle$ is defined as follows:

- $\mathcal{T}_{\mathsf{N}}(\!|\, C[\![t_1,...,t_n]\!] \,|\!) = \langle\, C[t'_1,...,t'_n] \,;\, \mathtt{true} \wedge \mathcal{C}ond_1 \wedge \cdots \wedge \mathcal{C}ond_n \,\rangle$
  where $\mathcal{T}_{\mathsf{N}}(\!|\, t_i \,|\!) = \langle\, t'_i \,;\, \mathcal{C}ond_i \,\rangle$ *for each* i,
- $\mathcal{T}_{\mathsf{N}}(\!|\, f(t_1,...,t_n) \,|\!) = \langle\, y \,;\, f^{\#}(y) \to tp_n(t'_1,...,t'_n) \wedge \mathcal{C}ond_1 \wedge \cdots \wedge \mathcal{C}ond_n \,\rangle$
  where y is a fresh variable and $\mathcal{T}_{\mathsf{N}}(\!|\, t_i \,|\!) = \langle\, t'_i \,;\, \mathcal{C}ond_i \,\rangle$ *for each* i.

It is clear that the procedure $\mathcal{T}_{\mathsf{N}}$ always terminates and $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$ is finite.

Example 4 *Consider the constructor TRS R_1 of Example 1. The natural inverse CTRS $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R_1, \{mult\})$ of R_1 is the following R_4 over $T(G, X)$:*

$$G_D = \{\, add^{\#}, mult^{\#}, add, mult \,\},\ \ G_C = \{\, s, 0, tp_0, tp_1, tp_2 \,\},$$

$$\begin{array}{rl} R_4 = \{ & add^{\#}(y) \to tp_2(0, y), \\ & add^{\#}(s(z)) \to tp_2(s(x), y) \ \Leftarrow\ add^{\#}(z) \to tp_2(x, y), \\ & mult^{\#}(0) \to tp_2(0, y), \\ & mult^{\#}(z) \to tp_2(s(x), y) \\ & \qquad\qquad \Leftarrow\ add^{\#}(z) \to tp_2(w, y) \wedge mult^{\#}(w) \to tp_2(x, y), \\ & add^{\#}(add(x, y)) \to tp_2(x, y),\ \ mult^{\#}(mult(x, y)) \to tp_2(x, y) \ \}. \end{array}$$

□

The third rule of R_4 shows that inverse CTRSs obtained by $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ are generally 4-CTRSs.

The generated inverse CTRS contains special rules like $f^{\#}(f(x_1, ..., x_n)) \to tp_n(x_1, ..., x_n)$, called *inverse rewrite rules*. These are not necessary if we assume that all functions are interpreted as strict. However, it is essential if the target reduction sequence contains reductions by erasing rules[1] and undefined values caused by partial functions. For example, consider the following TRS R_{g} and its natural inverse CTRS R'_{g}:

$$R_{\mathsf{g}} = \{\, g(x) \to mult(x, half(s(x))) \,\} \cup R_3 \cup R_1,$$

$$\begin{array}{rlll} R'_{\mathsf{g}} = \{ & g^{\#}(y) \to x \ \Leftarrow\ mult^{\#}(y) \to tp_2(x, z) \wedge half^{\#}(z) \to s(x), & & \\ & half^{\#}(0) \to 0, & half^{\#}(s(y)) \to s^2(x) \ \Leftarrow\ half^{\#}(y) \to x, & \\ & g^{\#}(g(x)) \to x, & half^{\#}(half(x)) \to x, & \} \cup R_4. \end{array}$$

Since $g(0) \xrightarrow{*}_{R_{\mathsf{g}}} 0$, it is expected that $g^{\#}(0) \xrightarrow{*}_{R'_{\mathsf{g}}} 0$. However, $g^{\#}(0)$ can't be rewritten to 0 without using the inverse rewrite rule $half^{\#}(half(x)) \to x$ since $half^{\#}(half(x)) \to x$ is necessary in order to satisfy the condition $half^{\#}(z) \to s(0)$.

[1] A rewrite rule $l \to r \ \Leftarrow\ Cond$ is called *erasing* if $\mathcal{V}ar(l) \supset \mathcal{V}ar(r)$.

It is easy to see the following properties on rules in the generated inverse CTRSs.

Proposition 1 *Let R be a constructor TRS over $T(F, X)$, $F_0 \subseteq F_D$ and $R' = \mathcal{InvCTRS}_{\mathsf{N}}(R, F_0)$. Each conditional rewrite rule in R', except inverse rewrite rules, is written as $f^{\#}(C[y_1, ..., y_m]) \to r \Leftarrow \bigwedge_{i=1}^{m} f_i^{\#}(y_i) \to r_i \wedge \bigwedge_{i=m+1}^{n} f_i^{\#}(z_i) \to r_i$, and satisfies the followings:*

- $\forall i, y_i \notin \mathcal{V}ar(r) \cup \mathcal{V}ar(C),\ \forall j, y_i \notin \mathcal{V}ar(r_j)$,
- $\forall i, z_i \notin \mathcal{V}ar(r) \cup \mathcal{V}ar(C),\ \exists k < i, (z_i \in \mathcal{V}ar(r_k)) \wedge (\ z_i$ *occurs once in* $r_k\ )$, $\forall j \neq k, z_i \notin \mathcal{V}ar(r_j)$.

If each rule $f(p_1, ..., p_n) \to r \in R$ with $f \in \mathcal{R}eq_{\mathsf{N}}(R, F_0)$ is right-linear, then each rule in R written as above also satisfies the following two properties:

- $\mathcal{V}ar(C) \cap \bigcup_{i=1}^{n} \mathcal{V}ar(r_i) = \emptyset$,
- $\forall i, j, (i \neq j) \Rightarrow (\mathcal{V}ar(r_i) \cap \mathcal{V}ar(r_j) = \emptyset$.

3.4 Correctness of $\mathcal{InvCTRS}_{\mathsf{N}}$

In this section, we prove the correctness of $\mathcal{InvCTRS}_{\mathsf{N}}$. For some substitution σ and σ', if $\mathcal{D}om(\sigma) \supseteq \mathcal{D}om(\sigma')$ then we write $\sigma \geqslant \sigma'$. Let F be a signature. We call σ a *constructor substitution* if $x\sigma \in T(F_C)$ for any $x \in \mathcal{D}om(\sigma)$, call σ a *ground substitution* if $x\sigma \in T(F)$ for any $x \in \mathcal{D}om(\sigma)$, and call σ a *normal form substitution* if $x\sigma \in T(F, X)$ is a n.f. for any $x \in \mathcal{D}om(\sigma)$.

In the following proposition, we show some properties of $\mathcal{T}_{\mathsf{N}}$.

Proposition 2 *Let R be a constructor TRS over $T(F, X)$, $F_0 \subseteq F_D$, $R' = \mathcal{InvCTRS}_{\mathsf{N}}(R, F_0)$, a term $t \in T(F, X)$ and $\mathcal{T}_{\mathsf{N}}[\![\, t \,]\!] = \langle\, t' ;\ \bigwedge_{i=1}^{k} s_i \to t_i \,\rangle$. (1) t' is a constructor term and $t \equiv t'\sigma$ for some substitution σ such that $\mathcal{D}om(\sigma) = \mathcal{V}ar(t') - \mathcal{V}ar(t)$, and (2) $\mathcal{T}_{\mathsf{N}}[\![\, t\theta \,]\!] = \langle\, t'\theta ;\ \bigwedge_{i=1}^{k} s_i\theta \to t_i\theta \,\rangle$ for any constructor substitution θ such that $\mathcal{D}om(\theta) = \mathcal{V}ar(t)$. In this case, we write $\bigwedge_{i=1}^{k} s_i\theta \to t_i\theta$ as $Cond \cdot \theta$ where $Cond = \bigwedge_{i=1}^{k} s_i \to t_i$, and (3) $Cond \cdot \theta(\sigma, \to) = Cond(\theta\sigma, \to)$. (4) $\mathcal{T}_{\mathsf{N}}[\![\, t \,]\!]$ satisfies the followings.*

- *If* $\mathcal{T}_{\mathsf{N}}(\!\![\ C[\![t_1,...,t_n]\!]\ ]\!\!) = \langle C[p_1,...,p_n]; Cond_1 \wedge \cdots \wedge Cond_n\rangle$ *where* $\mathcal{T}_{\mathsf{N}}(\!\![\ t_i\ ]\!\!) = \langle\, p_i\,;\, Cond_i\,\rangle$*, then* $(\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t)) \cap (\mathcal{V}ar(Cond_j) - \mathcal{V}ar(t)) = \emptyset$ *for every* i *and* j $(i \neq j)$*.*

- *If* $\mathcal{T}_{\mathsf{N}}(\!\![\ f(t_1,...,t_n)\ ]\!\!) = \langle\, y\,;\ f^{\#}(y) \to tp_n(p_1,...,p_n) \wedge Cond_1 \wedge \cdots \wedge Cond_n\,\rangle$ *where* $\mathcal{T}_{\mathsf{N}}(\!\![\ t_i\ ]\!\!) = \langle\ p_i\ ;\ Cond_i\ \rangle$*, then* $y \notin \mathcal{V}ar(Cond_i)$ *for every* i *and* $(\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t)) \cap (\mathcal{V}ar(Cond_j) - \mathcal{V}ar(t)) = \emptyset$ *for every* i *and* j $(i \ni j)$*.*

Proof. These are trivial by the construction of $\mathcal{T}_{\mathsf{N}}$. □

A redex t is *innermost* if every $u \lhd t$ is a n.f.. A reduction $C[s] \to C[t]$ is an *innermost reduction*, written as $C[s] \to^{\mathrm{in}} C[t]$, if s is an innermost redex. Let F be a signature, $\to \subseteq T(F,X) \times T(F,X)$ and $T', T'' \subseteq T(F,X)$. The restriction of $\to$ on T' and T'' is defined as $\to|_{T' \times T''} = \{\ (s,t) \mid s \in T', t \in T'', s \to t\ \}$.

Lemma 1 *Let* R *be a constructor 3-TRS over* $T(F,X)$ *with* $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\mathrm{in}}_R|_{T(F)\times NF_R(F)}$*,* $F_0 \subseteq F_D$*,* $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$*,* $t \in T(\mathcal{R}eq_{\mathsf{N}}(R,F_0) \cup F_C, X)$ *and* $\mathcal{T}_{\mathsf{N}}(\!\![\ t\ ]\!\!) = \langle\, p\,;\, Cond\,\rangle$*.*

1. *If* $t \xrightarrow{*}_R s$ *for some normal form* $s \in NF_R(\mathcal{R}eq_{\mathsf{N}}(R,F_0) \cup F_C, X)$*, then* $s \equiv p\sigma$ *and* $Cond(\sigma, \to_{R'})$ *for some n.f. substitution* σ *such that* $\mathcal{D}om(\sigma) = \mathcal{V}ar(Cond) - \mathcal{V}ar(t)$*.*

2. *If* $Cond(\sigma, \to_{R'})$ *for some n.f. substitution* σ *such that* $\mathcal{D}om(\sigma) = \mathcal{V}ar(Cond) - \mathcal{V}ar(t)$*, then* $t \xrightarrow{*}_R p\sigma$*.*

Proof. We prove the first claim by induction on the lexicographic combination of the number k of the reduction $t \xrightarrow{k}{}^{\mathrm{in}}_R s$ and the structure of t.

- Consider the case that $t \equiv C[\![t_1,...,t_n]\!]$. Since the claim trivially hold from $t \equiv p \equiv s$ if $n = 0$, we concentrate the subcase $n > 0$. Then, we have $p \equiv C[p_1,...,p_n]$ and $Cond = Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{N}}(\!\![\ t_i\ ]\!\!) = \langle p_i\,; Cond_i\rangle$.

  Since C is not □ and contains no defined symbols, we have $t \equiv C[\![t_1,...,t_n]\!] \xrightarrow{k}_R C[s_1,...,s_n] \equiv s$ and $t_i \xrightarrow{k_i}_R s_i$ for $k_i \leq k$. Thus, we have $s_i \equiv p_i\sigma_i$ and $Cond_i(\sigma_i, \to_{R'})$ for some n.f. substitution σ_i such that $\mathcal{D}om(\sigma_i) = \mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ by induction hypothesis.

Since the sets $\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ ($\subseteq \mathcal{V}ar(p_i)$) are pairwise disjoint by Proposition 2 (4), $\sigma = \sigma_1 \cup \cdots \cup \sigma_n$ is a n.f. substitution.

Therefore, we have $s \equiv C[s_1, ..., s_n] \equiv C[p_1\sigma_1, ..., p_n\sigma_n] \equiv p\sigma$ and $Cond_i(\sigma, \rightarrow_{R'})$ for all i, which conclude the proof of this case.

- In the case that $t \equiv f(t_1, ..., t_n)$, we have $p \equiv y$ and $Cond = f^{\#}(y) \rightarrow tp_n(p_1, ..., p_n) \wedge Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{N}}[\![t_i]\!] = \langle p_i ; Cond_i \rangle$.

  This reduction sequence can be represented as $t \equiv f(t_1, ..., t_n) \stackrel{k'}{\rightarrow}{}^{\text{in}}_{R} f(s_1, ..., s_n) \stackrel{k''}{\rightarrow}_R s$ where $k' + k'' = k$. Hence, we have $t_i \stackrel{k'_i}{\rightarrow}_R s_i$, $k'_i \leq k' < k$. Since s_i is a normal form from finiteness of k'_i, we have $s_i \equiv p_i\sigma_i$ and $Cond_i(\sigma_i, \rightarrow_{R'})$ for some n.f. substitution σ_i such that $\mathcal{D}om(\sigma_i) = \mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ by induction hypothesis.

  If $f(s_1, ..., s_n)$ is a normal form, then we have $s \equiv f(s_1, ..., s_n)$ and let $\sigma = \sigma_1 \cup \cdots \cup \sigma_n \cup \{ y \mapsto f(s_1, ..., s_n) \}$. Since $\mathcal{V}ar(Cond_1) - \mathcal{V}ar(t_1)$,..., $\mathcal{V}ar(Cond_n) - \mathcal{V}ar(t_n)$ and $\{y\}$ are pairwise disjoint by Proposition 2 (4), σ is a n.f. substitution. Hence, we have $s \equiv y\sigma$ and $Cond(\sigma, \rightarrow_{R'})$ holds since $f^{\#}(y\sigma) \equiv f^{\#}(f(s_1, ..., s_n)) \rightarrow_{R'} tp_n(s_1, ..., s_n) \equiv tp_n(p_1\sigma_1, ..., p_n\sigma_n) \equiv tp_n(p_1, ..., p_n)\sigma$ by $f^{\#}(f(x_1, ..., x_n)) \rightarrow tp_n(x_1, ..., x_n) \in R$ and $Cond_i(\sigma, \rightarrow_{R'})$ from $\sigma \geqslant \sigma_i$.

  Otherwise, $f(s_1, ..., s_n) \stackrel{k''}{\rightarrow}_R s$ contains a reduction $f(q_1, ..., q_n) \rightarrow r$ in R since $f(s_1, ..., s_n)$ is not a normal form. It's represented as $f(s_1, ..., s_n) \equiv f(q_1\theta, ..., q_n\theta) \rightarrow_R r\theta \stackrel{k''-1}{\rightarrow}_R s'\theta \equiv s$ for some n.f. substitution θ such that $\mathcal{D}om(\theta) = \bigcup_{i=1}^{n} \mathcal{V}ar(q_i)$. R' contains the rule $f^{\#}(r') \rightarrow tp_n(q_1, ..., q_n) \Leftarrow Cond'$ for $\mathcal{T}_{\mathsf{N}}[\![r]\!] = \langle r' ; Cond' \rangle$. We have $s' \equiv r'\sigma'$ and $Cond'(\sigma', \rightarrow_{R'})$ for some n.f. substitution σ' such that $\mathcal{D}om(\sigma') = \mathcal{V}ar(Cond') - \mathcal{V}ar(r)$ by inducton hypothesis. From $Cond'(\sigma', \rightarrow_{R'})$ and $\mathcal{V}ar(q_i) \cap \mathcal{D}om(\sigma') = \emptyset$, we have $f^{\#}(r'\sigma') \rightarrow_{R'} tp_n(q_1\sigma', ..., q_n\sigma') \equiv tp_n(q_1, ..., q_n)$. Now let $\sigma = \sigma_1 \cup \cdots \cup \sigma_n \cup \{y \mapsto r'\sigma'\theta\}$. Since $\mathcal{V}ar(Cond_1) - \mathcal{V}ar(t_1), ..., \mathcal{V}ar(Cond_n) - \mathcal{V}ar(t_n)$ and $\{y\}$ are pairwise disjoint by Proposition 2 (4), σ is a n.f. substitution. Then, we have $s \equiv s'\theta \equiv r'\sigma'\theta \equiv y\sigma$ and $Cond(\sigma, \rightarrow_{R'})$ holds since the following reduction holds:

$$\begin{aligned} f^{\#}(y\sigma) &\equiv f^{\#}(r'\sigma'\theta) \to_{R'} tp_n(q_1\theta, ..., q_n\theta) \\ &\equiv tp_n(s_1, ..., s_n) \equiv tp_n(p_1\sigma_1, ..., p_n\sigma_n) \equiv tp_n(p_1, ..., p_n)\sigma, \end{aligned}$$

and $Cond_i(\sigma, \to_{R'})$ from $\sigma \geqslant \sigma_i$.

We prove that $Cond(\sigma, \underset{k}{\to}_{R'})$ implies $t \xrightarrow{*}_R p\sigma$ for all k by induction on the lexicogrphic combination of the level k and the structre of t.

- Consider the case that $t \equiv C[\![t_1, ..., t_n]\!]$. Since the claim trivially hold from $t \equiv p$ if $n = 0$, we concentrate the subcase $n > 0$. Then, we have $p \equiv C[p_1, ..., p_n]$, $Cond = Cond_1 \wedge \cdots \wedge Cond_n$ and $Cond_i(\sigma, \underset{k}{\to}_{R'})$ where $\mathcal{T}_{\mathsf{N}}(\![\, t_i \,]\!) = \langle\, p_i \,;\, Cond_i \,\rangle$. Since $t_i \xrightarrow{*}_R p_i\sigma$ by induction hypothesis, we have $t \equiv C[\![t_1, ..., t_n]\!] \xrightarrow{*}_R C[p_1\sigma, ..., p_n\sigma] \equiv C[p_1, ..., p_n]\sigma \equiv p\sigma$.

- In the case that $t \equiv f(t_1, ..., t_n)$, we have $p \equiv y$ and $Cond = f^{\#}(y) \to tp_n(p_1, ..., p_n) \wedge Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{N}}(\![\, t_i \,]\!) = \langle\, p_i \,;\, Cond_i \,\rangle$. Then, we have $f^{\#}(y\sigma) \underset{k}{\to}_{R'} tp_n(p_1\sigma, ..., p_n\sigma)$ since $y\sigma$ is a normal form. Hence, there is a rule $f^{\#}(r') \to tp_n(q_1, ..., q_n) \Leftarrow Cond' \in R'$ such that $r'\sigma'\theta \equiv y\sigma$, $p_i\sigma \equiv q_i\theta$ and $Cond'(\sigma', \underset{k-1}{\to}_{R'})$ for some n.f. substitution σ' such that $\mathcal{D}om(\sigma') = \mathcal{V}ar(Cond') - \bigcup_{i=1}^{n} \mathcal{V}ar(q_i)$ and some n.f. substitution θ such that $\mathcal{D}om(\theta) = \bigcup_{i=1}^{n} \mathcal{V}ar(q_i)$. By construction of R', we have $f(q_1, ..., q_n) \to r \in R$ and $\mathcal{T}_{\mathsf{N}}(\![\, r \,]\!) = \langle r'; Cond' \rangle$. We have $r \xrightarrow{*}_R r'\sigma'$ by induction hypothesis.

  On the other hand, we have $t_i \xrightarrow{*}_R p_i\sigma$ from $Cond_i(\sigma, \underset{k-1}{\to}_{R'})$ by induction hypothesis.

  Therefore, $t \equiv f(t_1, ..., t_n) \xrightarrow{*}_R f(p_1\sigma, ..., p_n\sigma) \equiv f(q_1\theta, ..., q_n\theta) \to_R r\theta \xrightarrow{*}_R r'\sigma'\theta \equiv y\sigma$. □

Finally, we obtain the correctness of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ as in Theorem 1.

Theorem 1 *Let R be a constructor 3-TRS over $T(F, X)$ with $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*\,\mathrm{in}}_R|_{T(F)\times NF_R(F)}$, $F_0 \subseteq F_D$ and $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$. For all $f \in \mathcal{R}eq_{\mathsf{N}}(R, F_0)$ and all normal forms $t_1, ..., t_n, s \in NF_R(\mathcal{R}eq_{\mathsf{N}}(R, F_0) \cup F_C, X)$, $f(t_1, ..., t_n) \xrightarrow{*}_R s$ if and only if $f^{\#}(s) \to_{R'} tp_n(t_1, ..., t_n)$.*

Proof. It is trivial by letting $t \equiv f(t_1, ..., t_n)$, $p \equiv y$ and $Cond = f^{\#}(y) \to tp_n(p_1, ..., p_n) \wedge Cond_1 \wedge \cdots \wedge Cond_n$ in Lemma 1 where $\mathcal{T}_{\mathsf{N}}(\![\, t_i \,]\!) = \langle p_i; Cond_i \rangle$. □

From this theorem, the CTRS $\mathcal{InvCTRS}_{\mathsf{N}}(R, F_0)$ for a constructor 3-TRS R over $T(F, X)$ with $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\text{in}}_R|_{T(F)\times NF_R(F)}$ and $F_0 \subseteq F_D$ is the natural inverse CTRS of R in the sense of Definition 1.

Lemma 2 *Let R be a CTRS over $T(F, X)$. If R is ground-convergent then* $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\text{in}}_R|_{T(F)\times NF_R(F)}$.

Proof. This can be proved by Noetherian induction on the maximum number of the reduction. □

Corollary 1 *Let R be a ground-convergent constructor 3-TRS over $T(F, X)$, $F_0 \subseteq F_D$ and $R' = \mathcal{InvCTRS}_{\mathsf{N}}(R, F_0)$ over $T(G, X')$. For all $f \in \mathcal{R}eq_{\mathsf{N}}(R, F_0)$ and all normal forms $t_1, ..., t_n, s \in NF_R(\mathcal{R}eq_{\mathsf{N}}(R, F_0) \cup F_C, X)$, $f(t_1, ..., t_n) \xrightarrow{*}_R s$ if and only if $f^{\#}(s) \to_{R'} tp_n(t_1, ..., t_n)$.*

Proof. It is trivial by Theorem 1 and Lemma 2. □

Even if the input constructor TRS is confluent, the natural inverse CTRS of the input is not always confluent.

Example 5 *Consider $mult^{\#}(s^2(0))$ on R_4 of Example 4. We have $mult^{\#}(s^2(0))$ $\to_{R_4} tp_2(s(0), s^2(0))$ and $mult^{\#}(s^2(0)) \to_{R_4} tp_2(s^2(0), s(0))$, which show that $mult^{\#}$ is not confluent.* □

Chapter 4

Indexed Inverse CTRSs

In this chapter, we extend $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ to $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$ that generates indexed inverse CTRSs.

We introduce indexes in order to display the arguments to be fixed. Let f be a defined symbol with $arity(f) = n$. I is an *index* of f if $I \subseteq \{1, ..., n\}$. $|I|$ denotes the number of elements in I. We use I_k for representing the k-th smallest element of I where $1 \leq k \leq |I|$. t_{I_k} denotes a term of I_k-th argument of $f(t_1, ..., t_n)$. We use I, J to represent indexes. For an index I of f, $\hat{I}$ denotes the index containing all elements not in I ; $I \oplus \hat{I} = \{1, ..., arity(f)\}$. We use $f_I^{\#}$ as a defined symbol that computes a tuple of values of all $\hat{I}_j$-th arguments of f ; $f(t_1, ..., t_n) \xrightarrow{*} s$ implies $f_I^{\#}(s, tp_{|I|}(t_{I_1}, ..., t_{I_{|I|}})) \xrightarrow{*} tp_{|\hat{I}|}(t_{\hat{I}_1}, ..., t_{\hat{I}_{|\hat{I}|}})$. We call $f_I^{\#}$ an *indexed inverse defined symbol*, and also call f_I an *indexed defined symbol*. For a signature F, $F_{D_{\mathsf{I}}}$ denotes a set of all indexed defined symbols of F_D ; $F_{D_{\mathsf{I}}} = \{ f_I \mid f \in F_D, I \subseteq \{1, ..., arity(f)\} \}$.

Example 6 *Consider the function symbol mult in Example 1. The function symbol* $mult^{\#}_{\{2\}}$ *represents an inverse of mult that computes* $mult^{\#}_{\{2\}}(s^l(0), s^n(0)) \xrightarrow{*} s^m(0)$, *where* $mult(s^m(0), s^n(0)) \xrightarrow{*}_{R_1} s^l(0)$. □

4.1 Definition of Indexed Inverse CTRSs

In the following, we give the definition of indexed inverse CTRSs.

Definition 4 *Let R be a CTRS over $T(F,X)$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ and R' be a CTRS over $T(G,X)$ such that $G_C = F_C \cup \{tp_i \mid 0 \leq i \leq n\}$ and $G_D = F_D \oplus G'_D$, where n is the maximum number of arity of $f \in F_D$. R' is an indexed inverse of R with respect to $F_{0_{\mathsf{I}}}$ if for all $f_I \in F_{0_{\mathsf{I}}}$ there exists $g \in G'_D$ such that $f(t_1,...,t_n) \xrightarrow{*}_R s$ implies $g(s, tp_{|I|}(t_{I_1},...,t_{I_{|I|}})) \xrightarrow{*}_{R'} tp_{|\hat{I}|}(t_{\hat{I}_1},...,t_{\hat{I}_{|\hat{I}|}})$ for all $t_1,...,t_n, s \in T(F_C)$. In this case, we say that g is an indexed inverse function of f with respect to I.*

Since $f^{\#}_{\emptyset}$ is a natural inverse function of f, this definition is an extension of Definition 1.

4.2 Generation of Indexed Inverse CTRSs

Next, we give a generation of indexed inverse CTRSs. Firstly, we define the set $\mathcal{SV}ar(f(p_1,...,p_n) \to r, I)$ of variables to be solved, for a rule $f(p_1,...,p_n) \to r$ and an index I of f. We also define the index $\mathcal{I}ndex(f(t_1,...,t_n), X)$. Letting X be a set of variables to be solved, $i \in \mathcal{I}ndex(f(t_1,...,t_n), X)$ means that t_i contains no variables in X.

Definition 5 *Let R be a constructor TRS over $T(F,X)$. For a rule $l \to r \in R$ and an index I of f, $\mathcal{SV}ar(l \to r, I)$ is defined as follows:*

$$\mathcal{SV}ar(f(p_1,...,p_n) \to r, I) = \textstyle\bigcup_{i\in\hat{I}} \mathcal{V}ar(p_{\hat{I}_i}) - \bigcup_{i\in I} \mathcal{V}ar(p_{I_i}) - W,$$
$$\text{where } W = \emptyset \text{ if } top(r) \in F_D,\ W = \mathcal{V}ar(C) \text{ if } r \equiv C[\![r_1,...,r_n]\!].$$

For a defined symbol $f \in F_D$, terms $t_1,...,t_n \in T(F,X)$ and a set X' of variables, $\mathcal{I}ndex(f(t_1,...,t_n), X')$ is defined as follows:

$$\mathcal{I}ndex(f(t_1,...,t_n), X') = \{\, i \mid \mathcal{V}ar(t_i) \cap X' \neq \emptyset \,\}.$$

Example 7 *Consider $mult(s(x),y) \to add(mult(x,y),y)$ in R_1 in Example 1, then we have the followings:*

$$\mathcal{SV}ar(mult(s(x),y) \to add(mult(x,y),y), \{1\}) = \{y\},$$

$$\mathcal{I}ndex(add(mult(x,y),y), \{y\}) = \emptyset, \qquad \mathcal{I}ndex(mult(x,y), \{y\}) = \{1\}. \qquad \square$$

Next, we define $\mathcal{I}term(t, X')$ which attaches an index to each defined symbol f in t so that the i-th argument of f_I contains no variables in X' for each $i \in I$.

Definition 6 *Let a term $t \in T(F, X)$ and $X' \subseteq X$. The term $\mathcal{I}term(t, X')$ over $T(F_{D_\mathsf{I}} \cup F_C, X)$ is defined as follows:*

- $\mathcal{I}term(C[\![t_1, ..., t_n]\!], X') = C[t'_1, ..., t'_n]$ *where* $\mathcal{I}term(t_i, X') = t'_i$,
- $\mathcal{I}term(f(t_1, ..., t_n), X') = f_I(t'_1, ..., t'_n)$,
  where $I = \mathcal{I}ndex(f(t_1, ..., t_n), X')$ *and* $\mathcal{I}term(t_i, X') = t'_i$.

Example 8 *Consider $mult(s(x), y) \to add(mult(x, y), y)$ in R_1 in Example 1 again. For the right-hand side $add(mult(x, y), y)$ of that, we have the followings:*

$$\mathcal{I}term(add(mult(x, y), y), \{y\}) = add_\emptyset(mult_{\{1\}}(x, y), y).$$

This means that both of arguments of add contain the variable y and only the second argument of mult contains y. In really, $mult^\#_{\{1\}}(x, y)$ means $mult_x(y)$ stated in Chapter 1. □

Secondly, we extend $\mathcal{R}eq_\mathsf{N}$ to $\mathcal{R}eq_\mathsf{I}$, and $\mathcal{I}nv\mathcal{CTRS}_\mathsf{N}$ to $\mathcal{I}nv\mathcal{CTRS}_\mathsf{I}$ that generates indexed inverse CTRSs.

Definition 7 *Let R be a constructor TRS over $T(F, X)$ and $F_{0_\mathsf{I}} \subseteq F_{D_\mathsf{I}}$. We define $\mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}})$ to be the smallest set satisfying the followings:*

- $\mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}}) \supseteq F_{0_\mathsf{I}}$,
- $\mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}}) \supseteq \{\, h_I \mid l \to r \in R, top(l) = f_J \in \mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}}),$
  $h_I \mathrel{\dot{\in}} \mathcal{I}term(r, \mathcal{SV}ar(l \to r, J)) \,\}$.

Example 9 *For R_1 of Example 1, $\mathcal{R}eq_\mathsf{I}(R_1, \{mult_{\{1\}}\}) = \{mult_{\{1\}}, add_\emptyset\}$. This means that $mult^\#_{\{1\}}$ and $add^\#_\emptyset$ are necessary to compute $mult^\#_{\{1\}}$.* □

For a term $t \in T(F, X)$, $\bar{t}$ denotes a term obtained from t by replacing each defined symbol f in t with $\overline{f} \notin F_D$. For example, consider the case of $t \equiv add(mult(x, y), y)$. Then, $\bar{t}$ represent $\overline{add}(\overline{mult}(x, y), y)$.

Definition 8 *Let R be a constructor TRS over $T(F, X)$ and $F_{0_\mathsf{I}} \subseteq F_{D_\mathsf{I}}$ The signature G $(= G_D \oplus G_C)$ is defined as follows:*

$$G_D = \{\, f^\#_I, f \mid f_I \in \mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}}) \,\} \cup \{\, \overline{f} \mid f_I \in \mathcal{R}eq_\mathsf{I}(R, F_{0_\mathsf{I}}), |I| = arity(f) \,\},$$

$$G_C = F_C \cup \{\, tp_i \mid 0 \le i \le \max_{f \in F_D}(arity(f)) \,\}.$$

The indexed inverse CTRS $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$ *of* R *with respect to* $F_{0_{\mathsf{I}}}$ *is defined over* $T(G, X')$ *as follows:*

- $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$

$$\begin{aligned} &= \{\, \mathcal{I}nv\mathcal{R}ule_{\mathsf{I}}(l \to r, I) \mid l \to r \in R, top(l) = f, f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}) \,\} \\ &\quad \cup \{\ \ f_I^{\#}(f(x_1, ..., x_n), tp_{|I|}(x_{I_1}, ..., x_{I_{|I|}})) \to tp_{|\hat{I}|}(x_{\hat{I}_1}, ..., x_{\hat{I}_{|\hat{I}|}}) \\ &\qquad\qquad \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}) \,\} \\ &\quad \cup \{\ \overline{l} \to \overline{r} \mid\ \ l \to r \in R, top(l) = f, \\ &\qquad\qquad f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}), |I| = arity(f) \,\} \\ &\quad \cup \{\ \ f_I^{\#}(\overline{f}(x_1, ..., x_n), tp_{|I|}(x_{I_1}, ..., x_{I_{|I|}})) \to tp_{|\hat{I}|}(x_{\hat{I}_1}, ..., x_{\hat{I}_{|\hat{I}|}}) \\ &\qquad\qquad \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}), |I| = arity(f) \,\}, \end{aligned}$$

- $\mathcal{I}nv\mathcal{R}ule_{\mathsf{I}}(f(p_1, ..., p_n) \to r, I)$

$$\begin{aligned} &= f_I^{\#}(C[y_1, ..., y_m], tp_{|I|}(p_{I_1}, ..., p_{I_{|I|}})) \to tp_{|\hat{I}|}(p_{\hat{I}_1}, ..., p_{\hat{I}_{|\hat{I}|}}) \\ &\qquad \Leftarrow\ s_1 \to y_1 \wedge \cdots \wedge s_m \to y_m \wedge \mathcal{C}ond, \end{aligned}$$

where y_i*'s are fresh variables,* $C \in T(F_C, X')$ *and* $\mathcal{T}_{\mathsf{I}}(\!\!|\, \mathcal{I}term(r, \mathcal{SV}ar(f(p_1, ..., p_n) \to r, I)) \,|\!\!) = \langle\, C[\![s_1, ..., s_m]\!]\,;\, \mathcal{C}ond \,\rangle$.

The procedure $\mathcal{T}_{\mathsf{I}}$ *is defined as follow:*

- $\mathcal{T}_{\mathsf{I}}(\!\!|\, C[\![t_1, ..., t_n]\!] \,|\!\!) = \langle\, C[t'_1, ..., t'_n]\,;\, \mathtt{true} \wedge \mathcal{C}ond_1 \wedge \cdots \wedge \mathcal{C}ond_n \,\rangle$ *where* $\mathcal{T}_{\mathsf{I}}(\!\!|\, t_i \,|\!\!) = \langle\, t'_i\,;\, \mathcal{C}ond_i \,\rangle$ *for each* i *and* $C \not\equiv \Box$,

- $\mathcal{T}_{\mathsf{I}}(\!\!|\, f_I(t_1, ..., t_n) \,|\!\!)$

$$\begin{cases} = \langle\, y\,;\, f_I^{\#}(y, tp_{|I|}(t'_{I_1}, ..., t'_{I_{|I|}})) \to tp_{|\hat{I}|}(t'_{\hat{I}_1}, ..., t'_{\hat{I}_{|\hat{I}|}}) \wedge \bigwedge_{i=1}^{n} \mathcal{C}ond_i \,\rangle, \\ \qquad \text{where } |I| < arity(f),\ y \text{ is a fresh variable} \\ \qquad \text{and } \mathcal{T}_{\mathsf{I}}(\!\!|\, t_i \,|\!\!) = \langle\, t'_i\,;\, \mathcal{C}ond_i \,\rangle, \\ = \langle\, \overline{f}(t'_1, ..., t'_n)\,;\, \mathtt{true} \,\rangle \\ \qquad \text{where } |I| = arity(f) \text{ and } \mathcal{T}_{\mathsf{I}}(\!\!|\, t_i \,|\!\!) = \langle\, t'_i\,;\, \mathtt{true} \,\rangle. \end{cases}$$

It is clear that the procedure $\mathcal{T}_{\mathsf{I}}$ always terminates and $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_0)$ is finite. We abbreviate $f_\emptyset^{\#}(t, tp_0)$ to $f_\emptyset^{\#}(t)$ if that causes no confusion.

Example 10 *Consider the indexed inverses of mult of* R_1 *in Example 1.* $R_5 = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R_1, \{mult_{\{1\}}, mult_{\{2\}}, mult_\emptyset\})$ *is shown in Figure 4.1.* □

$$
\begin{array}{ll}
R_5 = \{ & add^{\#}_{\emptyset}(y) \to tp_2(0,y), \quad add^{\#}_{\emptyset}(add(x,y)) \to tp_2(x,y),\\
& add^{\#}_{\emptyset}(s(z)) \to tp_2(s(x),y) \Leftarrow add^{\#}_{\emptyset}(z) \to tp_2(x,y),\\
& mult^{\#}_{\{1\}}(0,0) \to y, \quad mult^{\#}_{\{1\}}(mult(x,y),x) \to y,\\
& mult^{\#}_{\{1\}}(z,s(x)) \to y \Leftarrow add^{\#}_{\emptyset}(z) \to tp_2(w,y) \wedge mult^{\#}_{\{1\}}(w,x) \to y,\\
& add^{\#}_{\{2\}}(y,y) \to 0, \quad add^{\#}_{\{2\}}(add(x,y),y) \to x,\\
& add^{\#}_{\{2\}}(s(z),y) \to s(x) \Leftarrow add^{\#}_{\{2\}}(z,y) \to x,\\
& mult^{\#}_{\{2\}}(0,y) \to 0, \quad mult^{\#}_{\{2\}}(mult(x,y),y) \to x,\\
& mult^{\#}_{\{2\}}(z,y) \to s(x) \Leftarrow add^{\#}_{\{2\}}(z,y) \to w \wedge mult^{\#}_{\{2\}}(w,y) \to x,\\
& mult^{\#}_{\emptyset}(0) \to tp_2(0,y), \quad mult^{\#}_{\emptyset}(mult(x,y),tp_0) \to tp_2(x,y),\\
& mult^{\#}_{\emptyset}(z) \to tp_2(s(x),y) \Leftarrow add^{\#}_{\emptyset}(z) \to tp_2(w,y) \wedge mult^{\#}_{\emptyset}(w) \to tp_2(x,y) \quad \}.
\end{array}
$$

Figure 4.1: Indexed inverse CTRS of *mult*.

Note that the indexed inverse CTRSs obtained by $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$ are also generally 4-CTRSs.

4.3 Correctness of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$

In this section, we prove the correctness of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$. In the following proposition, we show some properties of $\mathcal{T}_{\mathsf{I}}$. Letting a signature F and a substitution σ such that $x\sigma \in T(F,X)$ for every $x \in \mathcal{D}om(\sigma)$, $\overline{\sigma}$ means that $\mathcal{D}om(\overline{\sigma}) = \mathcal{D}om(\sigma)$ and $x\overline{\sigma} = \overline{t}$ for every $x\sigma = t$.

Proposition 3 *Let R be a constructor TRS over $T(F,X)$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ and $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$. Let a term $t \in T(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X)$ such that $\mathcal{I}term(t, X' \subseteq \mathcal{V}ar(t)) \dot{\ni} f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$, and let $\mathcal{T}_{\mathsf{I}}[\![\, \mathcal{I}term(t,X') \,]\!] = \langle\, t' \,;\, \bigwedge_{i=1}^{k} s_i \to t_i \,\rangle$. (1) $t' \equiv \overline{u}$ and $t \equiv u\sigma$ for some $u \in T(F, X' \cup \mathcal{V}ar(Cond))$ and some substitution σ such that $\mathcal{D}om(\sigma) = X'$, and (2) $\mathcal{T}_{\mathsf{I}}[\![\, \mathcal{I}term(t,X')\theta \,]\!] = \langle\, t'\overline{\theta} \,;\, \bigwedge_{i=1}^{k} s_i\overline{\theta} \to t_i\overline{\theta} \,\rangle$ for any substitution θ such that $\mathcal{D}om(\theta) = \mathcal{V}ar(t)$, $x\theta \in T(F_C, X)$ for every $x \in X'$ and $x\theta \in T(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X)$ for $x \in (\mathcal{V}ar(t) - X')$. In this case, we also write $\bigwedge_{i=1}^{k} s_i\overline{\theta} \to t_i\overline{\theta}$ as $Cond \cdot \overline{\theta}$ where $Cond = \bigwedge_{i=1}^{k} s_i \to t_i$, and (3) $Cond \cdot \overline{\theta}(\sigma, \to_{R'}) = Cond(\overline{\theta}\sigma, \to_{R'})$. (4) $\mathcal{T}_{\mathsf{I}}[\![\, \mathcal{I}term(t,X') \,]\!]$ satisfies the followings.*

- *If* $top(\mathcal{I}term(t, X') \in F_C$, *then* $(\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t)) \cap (\mathcal{V}ar(Cond_j) - \mathcal{V}ar(t)) = \emptyset$ *for every* i *and* j $(i \neq j)$ *where* $t \equiv C[\![t_1, ..., t_n]\!]$,

  $$\mathcal{T}_{\mathsf{I}}(\![\ \mathcal{I}term(C[\![t_1, ..., t_n]\!], X')\ ]\!) = \langle\ C[u_1, ..., u_n]\ ;\ Cond_1 \wedge \cdots \wedge Cond_n\ \rangle$$

  and $\mathcal{T}_{\mathsf{I}}(\![\ \mathcal{I}term(t_i, X')\ ]\!) = \langle\ u_i\ ;\ Cond_i\ \rangle$.

- *If* $top(\mathcal{I}term(t, X')) = f_I$ *and* $|I| < arity(f)$, *then* $y \notin \mathcal{V}ar(Cond_i)$ *for every* i *and* $(\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t)) \cap (\mathcal{V}ar(Cond_j) - \mathcal{V}ar(t)) = \emptyset$ *for every* i *and* j $(i \ni j)$ *where* $t \equiv f(t_1, ..., t_n)$, $\mathcal{T}_{\mathsf{I}}(\![\ \mathcal{I}term(f(t_1, ..., t_n), X')\ ]\!) = \langle\ y\ ;\ f^{\#}(y, tp_{|I|}(u_{I_1}, ..., u_{I_{|I|}})) \to tp_{|\hat{I}|}(u_{\hat{I}_1}, ..., u_{\hat{I}_{|\hat{I}|}})) \wedge Cond_1 \wedge \cdots \wedge Cond_n\ \rangle$ *and* $\mathcal{T}_{\mathsf{I}}(\![\ \mathcal{I}term(t_i, X')\ ]\!) = \langle\ u_i\ ;\ Cond_i\ \rangle$.

Proof. These are trivial by the construction of $\mathcal{T}_{\mathsf{I}}$. □

Lemma 3 *Let* R *be a constructor TRS over* $T(F, X)$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ *and* $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$. *For all* $t, s \in T(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}), |I| = arity(f)\} \cup F_C, X)$, $t \xrightarrow{*}_R s$ *if and only if* $\overline{t} \xrightarrow{*}_{R'} \overline{s}$.

Proof. It is trivial by $\{\ \overline{l} \to \overline{r} \in R' \mid top(l) = f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}), |I| = arity(f)\ \}$ for $\{l \to r \in R \mid top(l) = f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}}), |I| = arity(f)\}$ from the construction of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$. □

Lemma 4 *Let* R *be a constructor 3-TRS over* $T(F, X)$ *with* $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\text{in}}_R|_{T(F)\times NF_R(F)}$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ *and* $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$. *Let* $t \in T(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X')$ *such that* $\mathcal{I}term(t, X') \dot{\ni} f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$, *and* $\mathcal{T}_{\mathsf{I}}(\![\ \mathcal{I}term(t, X')\ ]\!) = \langle\ \overline{u}\ ;\ Cond\ \rangle$ *where* $u \in T(F, X' \cup \mathcal{V}ar(Cond))$.

1. *If* $t \xrightarrow{*}_R s$ *for some normal form* $s \in NF_R(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X)$, *then* $\overline{u}\,\overline{\sigma} \xrightarrow{*}_{R'} \overline{s}$ *and* $Cond(\overline{\sigma}, \to_{R'}\ )$ *for some n.f. substitution* σ *such that* $\mathcal{D}om(\sigma) = \mathcal{V}ar(Cond) - \mathcal{V}ar(t)$.
2. *If* $Cond(\overline{\sigma}, \to_{R'}\ )$ *for some n.f. substitution* σ *such that* $\mathcal{D}om(\sigma) = \mathcal{V}ar(Cond) - \mathcal{V}ar(t)$, *then* $t \xrightarrow{*}_R u\sigma$.

Proof. We prove the first claim by induction on the lexicographic combination of the number k of the reduction $t \xrightarrow{k}{}^{\text{in}}_R s$ and the structure of t.

- Consider the case that $t \equiv C[\![t_1,...,t_n]\!]$. Since the claim trivially hold from $t \equiv u \equiv s$ if $n = 0$, we concentrate the subcase $n > 0$. Then, we have $\mathcal{I}term(t, X') \equiv C[\mathcal{I}term(t_1, X'), ..., \mathcal{I}term(t_n, X')]$, $u \equiv C[u_1, ..., u_n]$ and $Cond = Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(t_i, X')\,]\!) = \langle\, \overline{u}_i\,;\, Cond_i\,\rangle$.

  Since C is not $\Box$ and contains no defined symbols, we have $t \equiv C[\![t_1,...,t_n]\!] \xrightarrow{k}_R C[s_1,...,s_n] \equiv s$ and $t_i \xrightarrow{k_i}_R s_i$ for $k_i \le k$. Thus, we have $\overline{u}_i\overline{\sigma}_i \xrightarrow{*}_{R'} \overline{s}_i$ and $Cond_i(\overline{\sigma}_i, \rightarrow_{R'})$ for some n.f. substitution σ_i such that $\mathcal{D}om(\sigma_i) = \mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ by induction hypothesis.

  Since the sets $\mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ ($\subseteq \mathcal{V}ar(u_i)$) are pairwise disjoint by Proposition 3 (4), $\sigma = \sigma_1 \cup \cdots \cup \sigma_n$ is a substitution.

  Therefore, we have the following:

  $$\overline{u}\,\overline{\sigma} \equiv C[\overline{u}_1\overline{\sigma}_1, ..., \overline{u}_n\overline{\sigma}_n] \xrightarrow{*}_{R'} C[\overline{s}_1, ..., \overline{s}_n] \equiv \overline{s}$$

  and $Cond_i(\overline{\sigma}, \rightarrow_{R'})$ for all i, which conclude the proof of this case.

- In the case that $top(\mathcal{I}term(t, X')) = f_I$ and $|I| < arity(f)$, we have $t \equiv f(t_1, ..., t_n)$, $\mathcal{I}term(t, X') \equiv f_I(\mathcal{I}term(t_1, X'), ..., \mathcal{I}term(t_n, X')))$, $u \equiv y$ and $Cond = f_I^{\#}(y, tp_{|I|}(\overline{u}_{I_1}, ..., \overline{u}_{I_{|I|}})) \rightarrow tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}}) \wedge Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(t_i, X')\,]\!) = \langle\, \overline{u}_i\,;\, Cond_i\,\rangle$. We also have $u_{\hat{I}_j} \equiv z_{\hat{I}_j}$ from $top(\mathcal{I}term(t_i, X')) = h_J$, $h \in F_D$ and $|J| < arity(h)$.

  This reduction sequence can be represented as $t \equiv f(t_1, ..., t_n) \xrightarrow{k'}{}^{\text{in}}_R f(s_1, ..., s_n) \xrightarrow{k''}_R s$ where $k' + k'' = k$. Hence, we have $t_i \xrightarrow{k'_i}_R s_i$, $k'_i \le k' < k$. Since s_i is a normal form from finiteness of k'_i, we have $\overline{u}_i\overline{\sigma}_i \xrightarrow{*}_{R'} \overline{s}_i$ and $Cond_i(\overline{\sigma}_i, \rightarrow_{R'})$ for some n.f. substitution σ_i such that $\mathcal{D}om(\sigma_i) = \mathcal{V}ar(Cond_i) - \mathcal{V}ar(t_i)$ by induction hypothesis. Especially, $\overline{u}_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \equiv z_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \equiv \overline{s}_{\hat{I}_j}$ since $\overline{s}_{\hat{I}_j}$ is also a normal form, $\sigma_{\hat{I}_j}$ is a n.f. substitution and $\overline{u}_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \equiv z_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \xrightarrow{*}_R \overline{s}_{\hat{I}_j}$.

  If $f(s_1, ..., s_n)$ is a normal form, then we have $s \equiv f(s_1, ..., s_n)$ and let $\sigma = \sigma_1 \cup \cdots \cup \sigma_n \cup \{\, y \mapsto f(s_1, ..., s_n)\,\}$. Since $\mathcal{V}ar(Cond_1) - \mathcal{V}ar(t_1)$,..., $\mathcal{V}ar(Cond_n) - \mathcal{V}ar(t_n)$ and $\{y\}$ are pairwise disjoint by Proposition 3 (4), σ is a n.f. substitution. Hence, we have $\overline{y}\,\overline{\sigma} \equiv \overline{f}(\overline{s}_1, ..., \overline{s}_n) \equiv \overline{s}$ and $Cond(\overline{\sigma}, \rightarrow_{R'})$ holds since the following reduction holds:

$$
\begin{aligned}
f_I^{\#}(y\overline{\sigma}, tp_{|I|}(\overline{u}_{I_1}\overline{\sigma}, ..., \overline{u}_{I_{|I|}}\overline{\sigma})) &\equiv f_I^{\#}(\overline{f}(\overline{s}_1, ..., \overline{s}_n), tp_{|I|}(\overline{u}_{I_1}\overline{\sigma}_1, ..., \overline{u}_{I_{|I|}}\overline{\sigma}_n)) \\
&\xrightarrow{*}_{R'} f_I^{\#}(\overline{f}(\overline{s}_1, ..., \overline{s}_n), tp_{|I|}(\overline{s}_{I_1}, ..., \overline{s}_{I_{|I|}})) \rightarrow_{R'} tp_{|\hat{I}|}(\overline{s}_{\hat{I}_1}, ..., \overline{s}_{\hat{I}_{|\hat{I}|}}) \\
&\equiv tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}\overline{\sigma}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}}\overline{\sigma}_{\hat{I}_{|\hat{I}|}}) \equiv tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}})\overline{\sigma},
\end{aligned}
$$

by $f_I^{\#}(\overline{f}(x_1, ..., x_n), tp_{|I|}(x_{I_1}, ..., x_{I_{|I|}})) \rightarrow tp_{|\hat{I}|}(x_{\hat{I}_1}, ..., x_{\hat{I}_{|\hat{I}|}}) \in R'$, $\overline{u}_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \equiv z_{\hat{I}_j}\overline{\sigma}_{\hat{I}_j} \equiv \overline{s}_{\hat{I}_j}$ and $Cond_i(\overline{\sigma}, \rightarrow_{R'})$ from $\sigma \geqslant \sigma_i$.

Otherwise, $f(s_1, ..., s_n) \xrightarrow{k''}_R s$ contains a reduction $f(q_1, ..., q_n) \rightarrow r$ in R since $f(s_1, ..., s_n)$ is not a normal form. It's represented as $f(s_1, ..., s_n) \equiv f(q_1\theta, ..., q_n\theta) \rightarrow_R r\theta \xrightarrow{k''-1}_R s$ for some n.f. substitution θ such that $\mathcal{D}om(\theta) = \bigcup_{i=1}^n \mathcal{V}ar(q_i)$. R' contains the following rule:

$$
\begin{aligned}
f_I^{\#}(C[y_1, ..., y_m], tp_{|I|}(q_{I_1}, ..., q_{I_{|I|}})) &\rightarrow tp_{|\hat{I}|}(q_{\hat{I}_1}, ..., q_{\hat{I}_{|\hat{I}|}}) \\
&\Leftarrow \textstyle\bigwedge_{i=1}^m \overline{r}_i \rightarrow y_i \wedge Cond',
\end{aligned}
$$

for $\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(r, \mathcal{SV}ar(f(q_1, ..., q_n) \rightarrow r, I)) \,]\!) = \langle\, C[\![\overline{r}_1, ..., \overline{r}_m]\!] \,;\, Cond' \,\rangle$.

Letting $s \equiv C[s'_1, ..., s'_m]$, we have $C[\![\overline{r}_1, ..., \overline{r}_m]\!]\overline{\sigma}' \xrightarrow{*}_{R'} C[\overline{s}'_1, ..., \overline{s}'_m] \equiv \overline{s}$ and $Cond'(\overline{\sigma}', \rightarrow_{R'})$ for some n.f. substitution σ' such that $\mathcal{D}om(\sigma') = \mathcal{V}ar(Cond') - \mathcal{V}ar(r)$ by induction hypothesis. We also have $\overline{r}_i\overline{\sigma}' \xrightarrow{*}_{R'} \overline{s}'_i$ by Lemma 3 since $top(r_i) \in F_D$ by the construction of $\mathcal{I}nv\mathcal{R}ule_{\mathsf{I}}$. Let $\sigma'' = \sigma' \cup \{y_1 \mapsto s'_1, ..., y_m \mapsto s'_m\}$. Since $y_i \notin \mathcal{D}om(\sigma')$, σ'' is a n.f. substitution. From $\overline{r}_i\overline{\sigma}' \xrightarrow{*}_{R'} \overline{s}'_i$, $Cond'(\overline{\sigma}', \rightarrow_{R'})$ and $\mathcal{V}ar(q_i) \cap \mathcal{D}om(\sigma') = \emptyset$, we have the following:

$$
\begin{aligned}
& f_I^{\#}(C[y_1, ..., y_m]\overline{\sigma}'', tp_{|I|}(q_{I_1}\overline{\sigma}'', ..., q_{I_{|I|}}\overline{\sigma}'')) \\
&\quad \equiv f_I^{\#}(C[\overline{s}'_1, ..., \overline{s}'_m], tp_{|I|}(q_{I_1}\overline{\sigma}', ..., q_{I_{|I|}}\overline{\sigma}')) \\
&\quad \equiv f_I^{\#}(C[\overline{s}'_1, ..., \overline{s}'_m], tp_{|I|}(q_{I_1}, ..., q_{I_{|I|}})) \\
&\quad \rightarrow_{R'} tp_{|\hat{I}|}(q_{\hat{I}_1}\overline{\sigma}'', ..., q_{\hat{I}_{|\hat{I}|}}\overline{\sigma}'') \equiv tp_{|\hat{I}|}(q_{\hat{I}_1}, ..., q_{\hat{I}_{|\hat{I}|}}).
\end{aligned}
$$

Now let $\sigma = \sigma_1 \cup \cdots \cup \sigma_n \cup \{y \mapsto C[s'_1, ..., s'_m]\}$. Since $\mathcal{V}ar(Cond_1) - \mathcal{V}ar(t_1)$,..., $\mathcal{V}ar(Cond_n) - \mathcal{V}ar(t_n)$ and $\{y\}$ are pairwise disjoint by Proposition 3 (4), σ is a n.f. substitution. Then, we have $\overline{y}\,\overline{\sigma} \equiv C[\overline{s}'_1, ..., \overline{s}'_m] \equiv \overline{s}$

and $Cond(\overline{\sigma},\ \rightarrow_{R'}\ )$ holds since the following reduction holds:

$$\begin{array}{l} f_I^{\#}(y\overline{\sigma}, tp_{|I|}(\overline{u}_{I_1}, ..., \overline{u}_{I_{|I|}})\overline{\sigma}) \\ \quad \equiv f_I^{\#}(C[\overline{s}'_1, ..., \overline{s}'_m], tp_{|I|}(\overline{u}_{I_1}\overline{\sigma}_{I_1}, ..., \overline{u}_{I_{|I|}}\overline{\sigma}_{I_{|I|}})) \\ \quad \xrightarrow{*}_{R'} f_I^{\#}(C[\overline{s}'_1, ..., \overline{s}'_m], tp_{|I|}(\overline{s}_{I_1}, ..., \overline{s}_{I_{|I|}})) \\ \quad \equiv f_I^{\#}(C[\overline{s}'_1, ..., \overline{s}'_m], tp_{|I|}(q_{I_1}\overline{\theta}, ..., q_{I_{|I|}}\overline{\theta})) \rightarrow_{R'} tp_{|\hat{I}|}(q_{\hat{I}_1}\overline{\theta}, ..., q_{\hat{I}_{|\hat{I}|}}\overline{\theta}) \\ \quad \equiv tp_{|\hat{I}|}(\overline{s}_{\hat{I}_1}, ..., \overline{s}_{\hat{I}_{|\hat{I}|}}) \equiv tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}\overline{\sigma}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}}\overline{\sigma}_{\hat{I}_{|\hat{I}|}}) \equiv tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}})\overline{\sigma}, \end{array}$$

and $Cond_i(\overline{\sigma},\ \rightarrow_{R'}\ )$ from $\sigma \geqslant \sigma_i$.

- In the case that $top(\mathcal{I}term(t, X')) = f_I$ and $|I| = arity(f)$, we have $u \equiv t$ and $Cond = \mathtt{true}$. Hence, $\overline{t} \xrightarrow{*}_{R'} \overline{s}$ from $t \xrightarrow{k}_R s$ by Lemma 3.

We prove that $Cond(\overline{\sigma},\ \underset{k}{\rightarrow}_{R'}\ )$ implies $t \xrightarrow{*}_R u\sigma$ for all k by induction on the lexicographic combination of the level k of the reduction $\underset{k}{\xrightarrow{*}}_{R'}$ and the structure of t.

- Consider the case that $t \equiv C[\![t_1, ..., t_n]\!]$. Since the claim trivially hold from $t \equiv u \equiv s$ if $n = 0$, we concentrate the subcase $n > 0$. Then, we have $\mathcal{I}term(t, X') \equiv C[\mathcal{I}term(t_1, X'), ..., \mathcal{I}term(t_n, X')]$, $u \equiv C[u_1, ..., u_n]$ and $Cond = Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{I}}[\![\ \mathcal{I}term(t_i, X')\ ]\!] = \langle\ \overline{u}_i\ ;\ Cond_i\ \rangle$.

  Since $t_i \xrightarrow{*}_R u_i\sigma$ by induction hypothesis, we have the following:

  $$t \equiv C[\![t_1, ..., t_n]\!] \xrightarrow{*}_R C[u_1\sigma, ..., u_n\sigma] \equiv u\sigma.$$

- In the case that $top(\mathcal{I}term(t, X')) = f_I$ and $|I| < arity(f)$, we have $t \equiv f(t_1, ..., t_n)$, $\mathcal{I}term(t, X') \equiv f_I(\mathcal{I}term(t_1, X'), ..., \mathcal{I}term(t_n, X')))$, $u \equiv y$ and $Cond = f_I^{\#}(y, tp_{|I|}(\overline{u}_{I_1}, ..., \overline{u}_{I_{|I|}})) \rightarrow tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}}) \wedge Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{I}}[\![\ \mathcal{I}term(t_i, X')\ ]\!] = \langle\ \overline{u}_i\ ;\ Cond_i\ \rangle$. Then, we have the following reduction since $y\overline{\sigma}$ is a normal form:

  $$f_I^{\#}(y\overline{\sigma}, tp_{|I|}(\overline{u}_{I_1}, ..., \overline{u}_{I_{|I|}})\overline{\sigma}) \underset{k}{\rightarrow}_{R'} tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}})\overline{\sigma},$$

  and $Cond_i(\overline{\sigma},\ \rightarrow_{R'}\ )$ for every i. Hence, there is the following rule:

  $$\begin{array}{r} f_I^{\#}(C[y_1, ..., y_m], tp_{|I|}(q_{I_1}, ..., q_{I_{|I|}})) \rightarrow tp_{|\hat{I}|}(q_{\hat{I}_1}, ..., q_{\hat{I}_{|\hat{I}|}}) \\ \Leftarrow\ \bigwedge_{i=1}^{m} \overline{r}_i \rightarrow y_i \wedge Cond' \in R' \end{array}$$

such that $C[y_1,...,y_m]\theta' \equiv C[y_1\theta',...,y_m\theta'] \equiv y\sigma$, $q_{I_j}\theta \equiv u_{I_j}\sigma$, $q_{\hat{I}_j}(\theta \cup \sigma') \equiv u_{\hat{I}_j}$, $\overline{r}_i(\overline{\theta} \cup \overline{\sigma}') \xrightarrow{*}_{R'} y_i\overline{\theta}'$ and $Cond'(\overline{\theta} \cup \overline{\sigma}', \underset{k-1}{\rightarrow}_{R'})$ for some n.f. substitutions θ, θ' and σ' such that $\mathcal{D}om(\theta) = \bigcup_{j\in I} \mathcal{V}ar(q_{I_j})$, $\mathcal{D}om(\theta') = \{y_1,...,y_m\}$ and $\mathcal{D}om(\sigma') = \bigcup_{j\in\hat{I}} \mathcal{V}ar(q_{\hat{I}_j}) - \bigcup_{j\in I} \mathcal{V}ar(q_{I_j})$. By construction of R', we have $f(q_1,...,q_n) \to r \in R$ and $\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(r, \mathcal{SV}ar(f(q_1,...,q_n) \to r, I))\,]\!) = \langle\, C[\![\overline{r}_1,...,\overline{r}_m]\!]\,;\, Cond' \,\rangle$. Moreover, we have the following:

$$\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(r, \mathcal{SV}ar(f(q_1,...,q_n) \to r, I))\theta \,]\!) = \langle\, C[\![\overline{r}_1,...,\overline{r}_m]\!]\overline{\theta}\,;\, Cond' \cdot \overline{\theta} \,\rangle$$

and $Cond' \cdot \overline{\theta}(\overline{\sigma}', \rightarrow_{R'}) = Cond'(\overline{\theta} \cup \overline{\sigma}', \rightarrow_{R'})$ by Proposition 3 (2), (3). We have $r\theta \xrightarrow{*}_R (C[\![r_1,...,r_m]\!]\theta)\sigma'$ by induction hypothesis. On the other hand, we have $t_i \xrightarrow{*}_R u_i\sigma$ from $Cond_i(\overline{\sigma}, \underset{k-1}{\rightarrow}_{R'})$ by induction hypothesis. We also have $r_i(\theta \cup \sigma') \xrightarrow{*}_R y_i\theta'$ from $\overline{r}_i(\overline{\theta} \cup \overline{\sigma}') \xrightarrow{*}_{R'} y_i\overline{\theta}'$ by Lemma 3. Therefore,

$$\begin{aligned} t \equiv f(t_1,...,t_n) &\xrightarrow{*}_R f(u_1\sigma,...,u_n\sigma) \equiv f(q_1(\theta \cup \sigma'),...,q_n(\theta \cup \sigma')) \\ &\rightarrow_R r(\theta \cup \sigma') \equiv (r\theta)\sigma' \xrightarrow{*}_R (C[\![r_1,...,r_m]\!]\theta)\sigma' \\ &\equiv C[\![r_1(\theta \cup \sigma'),...,r_m(\theta \cup \sigma')]\!] \xrightarrow{*}_R C[y_1\theta',...,y_m\theta'] \equiv y\sigma. \end{aligned}$$

- In the case that $top(\mathcal{I}term(t, X')\delta) = f_I$ and $|I| = arity(f)$, we have $t \equiv f(t_1,...,t_n)$, $\mathcal{I}term(t, X')\delta \equiv f_I(\mathcal{I}term(t_1, X')\delta,...,\mathcal{I}term(t_n, X')\delta))$, $u \equiv f(u_1,...,u_n)$ and $Cond = Cond_1 \wedge \cdots \wedge Cond_n$ where $\mathcal{T}_{\mathsf{I}}(\![\, \mathcal{I}term(t_i, X')\delta \,]\!) = \langle\, u_i\,;\, Cond_i \,\rangle$. We have $u_i\sigma \equiv t_i\delta$ by induction hypothesis. Therefore, $t\delta \equiv f(t_1\delta,...,t_n\delta) \xrightarrow{*}_R f(u_1\sigma,...,u_n\sigma) \equiv u\sigma$. □

We obtain the correctness of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$.

Theorem 2 *Let R be a constructor 3-TRS over $T(F, X)$ with $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*\,\mathrm{in}}_R|_{T(F)\times NF_R(F)}$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ and $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$. For all $f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$ and all normal forms $t_1,...,t_n, s \in NF_R(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X\})$, $f(t_1,...,t_n) \xrightarrow{*}_R s$ if and only if $f_I^{\#}(\overline{s}, tp_{|I|}(\overline{t}_{I_1},...,\overline{t}_{I_{|I|}})) \rightarrow_{R'} tp_{|\hat{I}|}(\overline{t}_{\hat{I}_1},..., \overline{t}_{\hat{I}_{|\hat{I}|}})$.*

Proof. It is trivial by letting $t \equiv f(t_1,...,t_n)$, $u \equiv y$ and $Cond = f_I^{\#}(y, tp_{|I|}(\overline{u}_{I_1}, ..., \overline{u}_{I_{|I|}})) \rightarrow_{R'} tp_{|\hat{I}|}(\overline{u}_{\hat{I}_1}, ..., \overline{u}_{\hat{I}_{|\hat{I}|}})$ in Lemma 4 where $\mathcal{T}_{\mathsf{I}}(\![\, t_i \,]\!) = \langle\, \overline{u}_i\,;\, Cond_i \,\rangle$. □

The CTRS obtained from R by $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$ is an indexed inverse CTRS of R in the sence of Definition 4.

Corollary 2 *Let R be a ground-convergent constructor 3-TRS over $T(F,X)$, $F_{0_{\mathsf{I}}} \subseteq F_{D_{\mathsf{I}}}$ and $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$. For all $f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})$ and all normal forms $t_1, ..., t_n, s \in NF_R(\{f \mid f_I \in \mathcal{R}eq_{\mathsf{I}}(R, F_{0_{\mathsf{I}}})\} \cup F_C, X\}$, $f(t_1, ..., t_n) \xrightarrow{*}_R s$ if and only if $f_I^{\#}(\overline{s}, tp_{|I|}(\overline{t}_{I_1}, ..., \overline{t}_{I_{|I|}})) \to_{R'} tp_{|\hat{I}|}(\overline{t}_{\hat{I}_1}, ..., \overline{t}_{\hat{I}_{|\hat{I}|}})$.*

Proof. It is trivial by Theorem 2 and Lemma 2. □

Corollary 3 *Let R be a ground-convergent constructor 3-TRS over $T(F,X)$, $F_0 \subseteq F_D$, $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$ and $R'' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}(R, \{ f_{\emptyset} \mid f \in F_0 \})$. For $f \in F_0$ and all normal forms $t_1, ..., t_n, s \in NF_R(\mathcal{R}eq_{\mathsf{N}}(R, F_0) \cup F_C, X\})$, $f^{\#}(s) \to_{R'} tp_n(t_1, ..., t_n)$ if and only if $f_{\emptyset}^{\#}(\overline{s}) \to_{R''} tp_n(\overline{t}_1, ..., \overline{t}_n)$.*

Proof. It is trivial from Theorem 1, 2. □

Chapter 5

Transformation of CTRSs to TRSs

E.Ohlebusch have proposed the transformation of deterministic 3-CTRSs to TRSs[Oh01]. In this chapter, we prove CTRSs obtained by our methods are deterministic 4-CTRSs, and also show that Ohlebusch's transformation works on deterministic 4-CTRSs.

5.1 Deterministic 4-CTRSs

The definition of deterministic 4-CTRSs is the same as for 3-CTRSs.

Definition 9 *Let R be an oriented 4-CTRS. A conditional rewrite rule $l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$ is called deterministic if $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l) \cup \bigcup_{j=1}^{i-1} \mathcal{V}ar(t_j)$ for every i. The CTRS R is called deterministic if every rewrite rule in R is deterministic.*

Theorem 3 *CTRSs obtained by $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ and $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$ are deterministic 4-CTRSs.*

Proof. It is trivial by the constructions of $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}$ and $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$. □

Moreover, we show the conditions so that our CTRSs are deterministic 3-CTRSs.

Theorem 4 *Let R be a constructor 3-TRS over $T(F,X)$, $F_0 \subseteq F_D$ and $F_{0_\mathsf{I}} \subseteq F_{D_\mathsf{I}}$. $\mathcal{I}nv\mathcal{CTRS}_\mathsf{N}(R,F_0)$ is a deterministic 3-CTRS if $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r)$ for each rule $l \to r \in R$. $\mathcal{I}nv\mathcal{CTRS}_\mathsf{I}(R,F_{0_\mathsf{I}})$ is also a deterministic 3-CTRS if all of the following properties are satisfied.*

- *For every $f_I \in \mathcal{R}eq_\mathsf{I}(R,F_{0_\mathsf{I}})$, $\bigcup_{i\in\hat{I}} \mathcal{V}ar(p_{\hat{I}_i}) \subseteq \mathcal{V}ar(r) \cup \bigcup_{i\in I} \mathcal{V}ar(p_i)$ for each rule $f(p_1,...,p_n) \to r \in R$.*

- *For every $f_I \in \mathcal{R}eq_\mathsf{I}(R,F_{0_\mathsf{I}})$ with $|I| = arity(f)$, $\mathcal{V}ar(r) \subseteq \bigcup_{i=1}^{n} \mathcal{V}ar(p_i)$ for each rule $f(p_1,...,p_n) \to r \in R$.*

Proof. It is trivial by the definitions of $\mathcal{I}nv\mathcal{CTRS}_\mathsf{N}$ and $\mathcal{I}nv\mathcal{CTRS}_\mathsf{I}$. □

5.2 Formalization of Transformation

Next, we explain Ohlebusch's transformation $\mathcal{U}(R)$[Oh01]. Letting R be a CTRS, we use a label for each rule in R, i.e., $\rho : l \to r \Leftarrow Cond$. Letting $X_1,...,X_n$ be sets of variables, we use $\langle\!\langle X_1,...,X_n \rangle\!\rangle$ as the unique ordered-list representation of the set $X_1 \cup \cdots \cup X_n$.

Definition 10 *Let R be a deterministic 4-CTRS over $T(F,X)$. For each conditional rewrite rule $\rho : l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$, we prepare k fresh defined symbols $U_1^\rho,...,U_k^\rho$ in the transformation. We transform $\rho : l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$ into a set $U(\rho)$ of $k+1$ rewrite rules as follows:*

$$
\begin{array}{rl}
U(\rho) = \{ & l \to U_1^\rho(s_1, \langle\!\langle \mathcal{V}ar(l) \rangle\!\rangle), \\
& U_1^\rho(t_1, \langle\!\langle \mathcal{V}ar(l) \rangle\!\rangle) \to U_2^\rho(s_2, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \rangle\!\rangle), \\
& U_2^\rho(t_2, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \rangle\!\rangle) \to U_3^\rho(s_3, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1), \mathcal{EV}ar(t_2) \rangle\!\rangle), \\
& \ddots \\
& U_k^\rho(t_k, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1), ..., \mathcal{EV}ar(t_{k-1}) \rangle\!\rangle) \to r \quad \}.
\end{array}
$$

Note that $\mathcal{U}(l \to r) = \{ l \to r \}$ for every rewrite rule $l \to r$ in R. A TRS $\mathcal{U}(R)$ is defined as $\mathcal{U}(R) = \bigcup_{\rho\in R} U(\rho)$.

Since $U(\rho)$ is not affected by extra variables appeared only in the right-hand side, the transformation also works on 4-CTRSs. Remark that although we use the notation $\langle\!\langle \ldots \rangle\!\rangle$ to eliminate redundant variables in the definition of $\mathcal{U}(\rho)$, it is essentially same with the original definition. For example, the original transformation generates the rules $\{ f(x,y) \to U_1(g(x),x,y),\ U_1(z,x,y) \to U_2(g(y),x,y,z),\ U_2(z,x,y,z) \to U_3(g(z),x,y,z,z), U_3(w,x,y,z,z) \to w \}$ from $f(x,y) \to w \Leftarrow g(x) \to z \wedge g(y) \to z \wedge g(z) \to w$. The occurrence of z in the fifth argument of U_3 is redundant. Since we sometimes abbreviate $tp_1(t)$ to t, $U^i_j(t, x_1, ..., x_n)$ must be rewritten by innermost reduction.

Example 11 *We obtain the following CTRS $R_6 = \mathcal{U}(R_4)$:*

$$
\begin{array}{rll}
R_6 = \{ & add^{\#}(y) \to tp_2(0,y), \quad mult^{\#}(0) \to tp_2(0,y), & \\
& add^{\#}(s(z)) \to U^1_1(add^{\#}(z), z), & \\
& U^1_1(tp_2(x,y), z) \to tp_2(s(x), y)), & \\
& mult^{\#}(z) \to U^2_1(add^{\#}(z), z), & \\
& U^2_1(tp_2(w,y), z) \to U^2_2(mult^{\#}(w), y, z, w), & \\
& U^2_2(tp_2(x,y), y, z, w) \to tp_2(s(x), y) & \}.
\end{array}
$$
$\square$

Finally, we prove the correctness of $\mathcal{U}$.

Theorem 5 *Let R be a deterministic 4-CTRS over $T(F,X)$ and $R' = \mathcal{U}(R)$. For all $t, s \in T(F,X)$, $t \xrightarrow{*}_R s$ if and only if $t \xrightarrow{*}_{R'} s$.*

Proof. $\Rightarrow$). We prove this direction by induction on the number n and the level m of the reduction $t \underset{m}{\overset{n}{\to}}_R s$. We suppose that the first rule applied is $\rho : l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$. The substitution σ such that $t \equiv l\sigma$ satisfies $s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k$, that is $s_i\sigma \underset{m'}{\to}_R t_i\sigma$ with $m' < m$. Then, the reduction is $t \equiv l\sigma \underset{m}{\to}_R r\sigma \overset{n-1}{\to}_R s$. We have $r\sigma \xrightarrow{*}_{R'} s$ and $s_i\sigma \xrightarrow{*}_{R'} t_i\sigma$ by induction hypothesis. Moreover, we have the following by the definition of $U(\rho)$:

$$
\begin{array}{rll}
R' \supseteq \{ & l \to U^\rho_1(s_1, \langle\!\langle \mathcal{V}ar(l) \rangle\!\rangle), & \\
& U^\rho_1(t_1, \langle\!\langle \mathcal{V}ar(l) \rangle\!\rangle) \to U^\rho_2(s_2, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \rangle\!\rangle), & \\
& \vdots & \\
& U^\rho_k(t_k, \langle\!\langle \mathcal{V}ar(l), \mathcal{EV}ar(t_1), ..., \mathcal{EV}ar(t_{k-1}) \rangle\!\rangle) \to r & \}.
\end{array}
$$

Therefore, the following reduction holds:

$$\begin{aligned} t \quad &\equiv l\sigma \to_{R'} U_1^\rho(s_1\sigma, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle\sigma) \xrightarrow{*}_{R'} U_1^\rho(t_1\sigma, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle\sigma) \\ &\to_{R'} U_2^\rho(s_2\sigma, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \,\rangle\!\rangle\sigma) \xrightarrow{*}_{R'} \cdots \\ &\xrightarrow{*}_{R'} U_k^\rho(t_k\sigma, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1), ..., \mathcal{EV}ar(t_{k-1}) \,\rangle\!\rangle\sigma) \to_{R'} r\sigma \xrightarrow{*}_{R'} s. \end{aligned}$$

$\Leftarrow$). We prove this direction by induction on the number n of the reduction $t \xrightarrow{n}_{R'} s$. We suppose that the first rule applied is $l \to r \in R'$ with $top(l) \in F_D$ and $t \equiv l\sigma$ for some substitution σ.

- If $top(r) \in F \cup X$, the reduction is $t \equiv l\sigma \to_{R'} r\sigma \xrightarrow{n-1}_{R'} s$. We have $r\sigma \xrightarrow{*}_R s$ by induction hypothesis. Moreover, $l \to r$ is also in R. Hence, $t \equiv l\sigma \to_R r\sigma \xrightarrow{*}_R s$.

- Otherwise, $r \equiv U_1^\rho(s_1, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle)$ and the following holds for some k:

$$\begin{aligned} R' \supseteq \{ \quad & U_1^\rho(t_1, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle) \to U_2^\rho(s_2, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \,\rangle\!\rangle), \\ & \vdots \\ & U_k^\rho(t_k, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1), ..., \mathcal{EV}ar(t_{k-1}) \,\rangle\!\rangle) \to r' \quad \}, \end{aligned}$$

  where $r' \in T(F, X)$. Moreover, we have $\rho : l \to r' \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$. We have $s_i\sigma \xrightarrow{n_i}_R t_i\sigma$ by the following reduction:

$$\begin{aligned} t \quad &\equiv l\sigma \to_{R'} U_1^\rho(s_1\sigma, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle\sigma) \xrightarrow{n_1}_{R'} U_1^\rho(t_1\sigma, \langle\!\langle\, \mathcal{V}ar(l) \,\rangle\!\rangle\sigma) \\ &\to_{R'} U_2^\rho(s_2\sigma, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1) \,\rangle\!\rangle\sigma) \xrightarrow{n_2}_{R'} \cdots \\ &\xrightarrow{n_k}_{R'} U_k^\rho(t_k\sigma, \langle\!\langle\, \mathcal{V}ar(l), \mathcal{EV}ar(t_1), ..., \mathcal{EV}ar(t_{k-1}) \,\rangle\!\rangle\sigma) \to_{R'} r'\sigma \xrightarrow{n'}_{R'} s, \end{aligned}$$

  where $n_i < n$ and $n' = n - k - 1 - \sum_{i=1}^{k} n_i$. Hence, $t \equiv l\sigma \to_R r'\sigma$. We also have $r'\sigma \xrightarrow{*}_R s$ by induction hypothesis. Therefore, $t \xrightarrow{*}_R s$. □

For a constructor TRS R over $T(F, X)$, $F_0 \subseteq F_D$ and $F_{0_\mathsf{I}} \subseteq F_{D_\mathsf{I}}$, the notations $\mathcal{I}nv\mathcal{TRS}_\mathsf{N}(R, F_0)$ and $\mathcal{I}nv\mathcal{TRS}_\mathsf{I}(R, F_{0_\mathsf{I}})$ represent $\mathcal{U}(\mathcal{I}nv\mathcal{CTRS}_\mathsf{N}(R, F_0))$ and $\mathcal{U}(\mathcal{I}nv\mathcal{CTRS}_\mathsf{I}(R, F_{0_\mathsf{I}}))$, respectively.

We have the following interesting property between the right-linearity of the input constructor TRS and the left-linearity of its natural inverse TRS.

Theorem 6 *Let R be a constructor TRS over $T(F, X)$ and $F_0 \subseteq F_D$. If every rule $l \to r \in R$ is right-linear for all $top(l) \in \mathcal{R}eq_\mathsf{N}(R, F_0)$ then $\mathcal{I}nv\mathcal{TRS}_\mathsf{N}(R, F_0)$ is left-linear.*

Proof. It is trivial by Proposition 1. □

Chapter 6

Termination of Inverse Systems

In this chapter, we discuss the input class whose natural inverse systems terminate. The depth of a term t is the number of symbols in the longest path of its tree representation. The notation $depth(t)$ denotes the depth of a term t. For example, $depth(x) = 1$ and $depth(f(g(a), b)) = 3$. The minimum depth $mindepth(t)$ is the number of constructors in the shortest path of t from top to a variable or a defined symbol, including its last symbol (variable or defined symbol). For example, $mindepth(c_1(f(x), c_2(s(s(y)), 0))) = 2$.

Let R be a constructor CTRS over $T(F, X)$ and $F_0 \subseteq F_D$. We say that R is *depth-increasing* with respect to F_0 if every rule $l \to r \Leftarrow Cond \in R$ with $top(l) \in \mathcal{R}eq_{\mathsf{N}}(R, F_0)$ satisfies all of the following conditions:

- $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r)$,
- $top(r) \notin F_D$,
- $depth(l) \leq mindepth(r) + 1$.

Note that the first condition $\mathcal{V}ar(l) \subseteq \mathcal{V}ar(r)$ implies that $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$ is a deterministic 3-CTRS. We show that the natural inverse TRS obtained by $\mathcal{I}nv\mathcal{TRS}_{\mathsf{N}}$ from depth-increasing ground-convergent constructor 3-TRSs is innermost terminating. The next definition is based on the well-known fact that if $\succ$ is a well-founded partial order which is closed under context, then the order $\succ_{\mathsf{st}} = (\succ \cup \rhd)_+$ is also well-founded.

Definition 11 ([Oh01]) *We say that deterministic 3-CTRS R over $T(F,X)$ is quasi-reductive if there is an extension F' of the signature F such that $F \subseteq F'$ and a reduction order $\succ$ on $T(F',X')$ which, for every rule $l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k \in R$, every substitution σ and every $0 \leq i < k$ satisfies:*

- *if $s_j\sigma \succeq t_j\sigma$ for every $1 \leq j \leq i$, then $l\sigma \succ_{\mathsf{st}} s_{i+1}\sigma$,*
- *if $s_j\sigma \succeq t_j\sigma$ for every $1 \leq j \leq k$, then $l\sigma \succ r\sigma$.*

Lemma 5 ([Oh01]) *Let R be a deterministic 3-CTRS. If R is quasi-reductive then $\mathcal{U}(R)$ is innermost terminating.*

We decide the order $\succeq$ between terms by the depth of terms.

Definition 12 *The order $\succeq$ between terms s and t is defined that $depth(s) \geq depth(t)$ if and only if $s \succeq t$.*

We prepare the following lemmata for using Lemma 5.

Lemma 6 *Let R be a depth-increasing constructor TRS over $T(F,X)$. For all $f \in F$ and all normal forms $t_1, ..., t_n, s \in NF_R(F,X)$, if $f(t_1, ..., t_n) \xrightarrow{*}_R s$ then $depth(t_i) \leq depth(s)$ for all i.*

Proof. It is trivial by the conditions of depth-increasing. □

Lemma 7 *Let R be a constructor 3-TRS over $T(F,X)$ with $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\text{in}}_R|_{T(F)\times NF_R(F)}$, $F_0 \subseteq F_D$, $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$ and R be depth-increasing with respect to F_0. For all $f \in \mathcal{R}eq_{\mathsf{N}}(R,F_0)$ and all normal forms $t_1, ..., t_n, s \in NF_R(\mathcal{R}eq_{\mathsf{N}}(R,F_0) \cup F_C, X)$, if $f^{\#}(s) \to_{R'} tp_n(t_1, ..., t_n)$ then $depth(f^{\#}(s)) \geq depth(tp_n(t_1, ..., t_n))$.*

Proof. The reduction $f(t_1, ..., t_n) \xrightarrow{*}_R s$ holds by Theorem 1 and $depth(t_i) \leq depth(s)$ holds for every i by Lemma 6. Hence, $depth(f^{\#}(s)) \geq depth(tp_n(t_1, ..., t_n))$. □

Lemma 8 *Let R be a constructor 3-TRS over $T(F,X)$ with $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\text{in}}_R|_{T(F)\times NF_R(F)}$, $F_0 \subseteq F_D$, $R' = \mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$ and R be depth-increasing with respect to F_0. Let $\mathcal{T}_{\mathsf{N}}[\![\, u \,]\!] = \langle\, p\,;\, Cond \,\rangle$ for a term u and $Cond = s_1 \to$*

$t_1 \wedge \cdots \wedge s_k \rightarrow t_k$. *For all n.f. substitution* σ *such that* $Cond(\sigma, \rightarrow_{R'})$ *and all* $f \in \mathcal{R}eq_{\mathsf{N}}(R, F_0)$, $depth(f^{\#}(p\sigma)) \geq depth(s_i\sigma)$ *for every* i. *Especially, if* $top(u) \in F_C$ *then* $depth(f^{\#}(p\sigma)) > depth(s_i\sigma)$.

Proof. We prove this by induction on the structure of terms.

- If $u \equiv C[\![u_1, ..., u_n]\!]$, then $\mathcal{T}_{\mathsf{N}}(\![C[\![u_1, ..., u_n]\!]]\!) = \langle\, C[p_1, ..., p_n]\,;\, \bigwedge_{i=1}^{n} Cond_i \,\rangle$ where $\mathcal{T}_{\mathsf{N}}(\![u_i]\!) = \langle p_i; Cond_i\rangle$. Suppose $Cond_i = \bigwedge_{j=1}^{k_i} s_{i,j} \rightarrow t_{i,j}$, $f^{\#}(p_i\sigma) \succeq s_{i,j}\sigma$ for every j by induction hypothesis since $Cond_i(\sigma, \rightarrow_{R'})$ holds. Hence, $depth(f^{\#}(C[p_1, ..., p_n]\sigma)) > depth(f^{\#}(p_i\sigma)) \geq depth(s_{i,j}\sigma)$ for every i and j.

- If $u \equiv g(u_1, ..., u_n)$, then we have the following:

  $$\mathcal{T}_{\mathsf{N}}(\![g(u_1, ..., u_n)]\!) = \langle\, y\,;\, g^{\#}(y) \rightarrow tp_n(p_1, ..., p_n) \wedge \bigwedge_{i=1}^{n} Cond_i \,\rangle$$

  where $\mathcal{T}_{\mathsf{N}}(\![u_i]\!) = \langle\, p_i\,;\, Cond_i \,\rangle$. Suppose $Cond_i = \bigwedge_{j=1}^{k_i} s_{i,j} \rightarrow t_{i,j}$, we have $depth(f^{\#}(p_i\sigma)) \geq depth(s_{i,j}\sigma)$ for every j by induction hypothesis since $Cond_i(\sigma, \rightarrow_{R'})$ holds. It is clear that $depth(f^{\#}(y\sigma)) \geq depth(g^{\#}(y\sigma))$ holds. And $depth(g^{\#}(y\sigma)) \geq depth(tp_n(p_1, ..., p_n)\sigma)$ also holds by Lemma 7. Therefore, $depth(f^{\#}(y\sigma)) \geq depth(g^{\#}(y\sigma)) \geq depth(tp_n(p_1, ..., p_n)\sigma)) \geq depth(f^{\#}(p_i\sigma)) \geq depth(s_{i,j}\sigma)$ for every i and j. $\Box$

Finally, we can obtain the following theorem.

Theorem 7 *Let* R *be a constructor 3-TRS over* $T(F, X)$ *with* $\xrightarrow{*}_R|_{T(F)\times NF_R(F)} = \xrightarrow{*}{}^{\mathrm{in}}_R|_{T(F)\times NF_R(F)}$, $F_0 \subseteq F_D$. *If* R *is depth-increasing with respect to* F_0 *then the natural inverse TRS* $\mathcal{I}nv\mathcal{TRS}_{\mathsf{N}}(R, F_0)$ *is innermost terminating.*

Proof. Let R' be $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{N}}(R, F_0)$. Thanks to Lemma 5, we may show that R' is quasi-reductive. Consider $f^{\#}(C[y_1, ..., y_m]) \rightarrow tp_n(p_1, ..., p_n) \Leftarrow s_1 \rightarrow \wedge \cdots \wedge s_k \rightarrow t_k \in R'$. For all constructor substitution σ such that $s_i\sigma \xrightarrow{*}_{R'} t_i\sigma$, $f^{\#}(C[y_1, ..., y_m]\sigma) \succeq s_i\sigma$ for every i, and $f^{\#}(C[y_1, ..., y_m]\sigma) \succ tp_n(p_1, ..., p_n)\sigma$ by Lemma 7, 8 and the definition of $\succeq$. Since $C \not\equiv \Box$ and C is not variable from the condition of depth-increasing, $f^{\#}(C[y_1, ..., y_m]\sigma) \succ s_i\sigma$ for every i. Hence, this rule satisfy the conditions of quasi-reductive.

On the other hand, consider $f^{\#}(x) \rightarrow tp_n(p_1, ..., p_n) \in R'$ where p_i is either x or a constructor. This rule also satisfies the conditions of quasi-reductive. It is clear that $f^{\#}(f(x_1, ..., x_n)) \succ tp_n(x_1, ..., x_n)$ for every inverse rewrite rule $f^{\#}(f(x_1, ..., x_n)) \rightarrow tp_n(x_1, ..., x_n)$. Hence, R' is quasi-reductive. □

Corollary 4 *Let R be a ground-convergent constructor 3-TRS over $T(F, X)$ and $F_0 \subseteq F_D$. If R is depth-increasing with respect to F_0 then the natural inverse TRS $\mathcal{InvTRS}_{\mathsf{N}}(R, F_0)$ is innermost terminating.*

Proof. It is trivial by Theorem 7 and Lemma 2. □

We will try to investigate new reduction relation, called *flexible reduction*, of 4-TRSs. Although 4-TRSs don't always terminate in general, we show that flexible reduction simulates the original reduction and terminates in some cases. Flexible reduction is a reduction such that,

- Prohibit the substitution to extra variables, that is, $x \notin \mathcal{D}om(\sigma)$ when $C[l\sigma] \rightarrow_R C[r\sigma]$ by $l \rightarrow r \in R$ for extra variables x.
- Use unification for matching like the narrowing.

Example 12 *Consider a depth-increasing TRS R_7:*

$$\begin{aligned} R_7 = \{ \quad & add(0, y) \rightarrow y, \quad add(s(x), y) \rightarrow s(add(x, y)), \\ & mult(0, y) \rightarrow 0, \quad mult(x, 0) \rightarrow 0, \\ & mult(s(x), s(y)) \rightarrow s(add(mult(x, s(y)), y)) \quad \}. \end{aligned}$$

Consider the following TRS R_8 obtained by $\mathcal{InvTRS}_{\mathsf{N}}(R_7, \{mult\})$.

$$\begin{aligned} R_8 = \{ \quad & add^{\#}(y) \rightarrow tp_2(0, y), \\ & add^{\#}(s(z)) \rightarrow U_1^1(add^{\#}(z), z), \;\; U_1^1(tp_2(x, y), z) \rightarrow tp_2(s(x), y), \\ & mult^{\#}(0) \rightarrow tp_2(0, y), \quad mult^{\#}(0) \rightarrow tp_2(x, 0), \\ & mult^{\#}(s(z)) \rightarrow U_1^2(add^{\#}(z), z), \\ & U_1^2(tp_2(w, y), z) \rightarrow U_2^2(mult^{\#}(w), y, z, w), \\ & U_2^2(tp_2(x, s(y)), y, z, w) \rightarrow tp_2(s(x), s(y)), \\ & add^{\#}(add(x, y)) \rightarrow tp_2(x, y), \quad mult^{\#}(mult(x, y)) \rightarrow tp_2(x, y) \}. \end{aligned}$$

The flexible reduction is terminating and can simulate the original reduction. Consider the flexible reduction on $mult^{\#}(s(0))$*:*

$$mult^{\#}(s(0)) \to_{R_8} U_1^2(add^{\#}(0), 0) \xrightarrow{*}_{R_8} U_1^2(tp_2(0,0), 0)$$
$$\to_{R_8} U_2^2(mult^{\#}(0), 0, 0, 0) \xrightarrow{*}_{R_8} U_2^2(tp_2(0, y), 0, 0, 0) \underset{y \mapsto s(0)}{\rightsquigarrow_{R_8}} tp_2(s(0), s(0))$$

Consider the term $mult^{\#}(s^4(0))$*. There are sixteen reductions from* $mult^{\#}(s^4(0))$ *to a normal form. Three reductions end in a constructor term ;* $tp_2(s(0), s^4(0))$*,* $tp_2(s^2(0), s^2(0))$ *and* $tp_2(s^4(0), s(0))$*. All of the rests end in non-constructor term that contains* U_2^2 *like* $U_2^2(tp_2(s(0), s(0)), s^2(0), s^3(0), s(0))$*.* □

Note that for some rule $\rho : l \to r \Leftarrow s_1 \to t_1 \wedge \cdots \wedge s_k \to t_k$, if some substitution σ don't satisfy the condition $s_i \to t_i$, that is $s_i\sigma \xrightarrow{*} u_i \not\equiv t_i\sigma$, then the function U_i^{ρ} in $U_i^{\rho}(u_i, ...)$ introduced by $U(\rho)$ never disappears.

Chapter 7

Application

By $\mathcal{I}nv\mathcal{TRS}_{\mathsf{I}}(R_1, \{add_{\{2\}}\})$, we obtain the definition of $add^{\#}_{\{2\}}$ that is equivalent to add_y^{-1} in Chapter 1. Hence, R_{gcd} can be transformed to the following TRS:

$$
\begin{array}{rl}
R''_{\mathsf{gcd}} = \{ & gcd(x,y) \to gcd(add^{\#}_{\{2\}}(x,y),y), \\
& gcd(x,0) \to x, \quad gcd(x,y) \to gcd(y,x) \Leftarrow x < y, \\
& add^{\#}_{\{2\}}(y,y) \to 0, \quad add^{\#}_{\{2\}}(add(x,y),y) \to x, \\
& add^{\#}_{\{2\}}(s(z),y) \to U^1_1(add^{\#}_{\{2\}}(z,y),y,z), \quad U^1_1(x,y,z) \to s(x) \quad \}.
\end{array}
$$

The definition of R''_{gcd} is essentially the same as that of R'_{gcd}. Hence, R''_{gcd} can also compute gcd.

Next, we consider how to solve equations by using inverse systems. For solving equations, we know some methods, using "Narrowing"[BN98, Sl74, La75], "Inversion Algorithm"[DM99] and so on. Consider the following function:

$$n(d,m,y) = d + 30m + 360y, \ (0 \le d < 30, 0 \le m < 12). \tag{7.1}$$

The function $n(d,m,y)$ computes the number of elapsed days from the date $(0,0,0)$ to the date (d,m,y) with a fixed year-length of 360 days, a simple scheme of 12 equal-length months and 0-based dates. Assume $n(d,m,y) =^? 400$. When we solve this equation by using Narrowing, we don't know whether Narrowing terminates. On the other hand, Inversion Algorithm[DM99] would find the desirable solution $d = 10, m = 1, y = 1$, but also find all undesirable solutions, such as $d = 40, m = 12, y = 0$ because the algorithm don't accept the conditions $0 \le d < 30$ and $0 \le m < 12$ without its extension. In the rest of this chapter, we show a way to generate inverse systems of deterministic constructor 3-CTRSs to eliminate the undesirable solutions of the equation (7.1). Since a deterministic constructor 3-CTRS R over $T(F, X)$ can be transformed to a constructor 3-TRS, we obtain the inverse system of R with respect to F_0 ($\subseteq F_D$) by $\mathcal{InvTRS}_{\mathsf{I}}(\mathcal{U}(R), F_0)$. A CTRS and TRS representation $R_9, \mathcal{U}(R_9)$ of n is as follows:

$$
\begin{array}{ll}
R_9 = R_{10} \cup R_7 & \\
\quad \cup\{ & n(x_d, x_m, x_y) \to add(add(x_d, mult(s^{30}(0), x_m)), mult(s^{360}(0), x_y)) \\
& \qquad \Leftarrow leq(0, x_d) \to \mathtt{T} \wedge less(x_d, s^{30}(0)) \to \mathtt{T} \\
& \qquad\quad \wedge leq(0, x_m) \to \mathtt{T} \wedge less(x_m, s^{12}(0)) \to \mathtt{T} \quad \},
\end{array}
$$

$$
\begin{array}{ll}
\mathcal{U}(R_9) = R_{10} \cup R_7 & \\
\quad \cup\{ & n(x_d, x_m, x_y) \to u_1(leq(0, x_d), x_d, x_m, x_y), \\
& u_1(\mathtt{T}, x_d, x_m, x_y) \to u_2(less(x_d, s^{30}(0)), x_d, x_m, x_y), \\
& u_2(\mathtt{T}, x_d, x_m, x_y) \to u_3(leq(0, x_m), x_d, x_m, x_y), \\
& u_3(\mathtt{T}, x_d, x_m, x_y) \to u_4(less(x_m, s^{12}(0)), x_d, x_m, x_y), \\
& u_4(\mathtt{T}, x_d, x_m, x_y) \to \\
& \qquad add(add(x_d, mult(s^{30}(0), x_m)), mult(s^{360}(0), x_y)) \quad \},
\end{array}
$$

$$
\begin{array}{lll}
R_{10} = \{ & leq(0, 0) \to \mathtt{T}, & less(0, 0) \to \mathtt{F}, \\
& leq(0, s(y)) \to \mathtt{T}, & less(0, s(y)) \to \mathtt{T}, \\
& leq(s(x), 0) \to \mathtt{F}, & less(s(x), 0) \to \mathtt{F}, \\
& leq(s(x), s(y)) \to leq(x, y), & less(s(x), s(y)) \to leq(x, y) \quad \},
\end{array}
$$

where leq and $less$ are defined symbols that represent $\leq$ and $<$, respectively. Let the indexed inverse TRS $R_{11} = \mathcal{InvTRS}_{\mathsf{I}}(\mathcal{U}(R_9), \{n_\emptyset\})$ (Figure 7.1). R_{11} can search only for the desirable solutions and terminates by flexible reduction since each function is terminating. Thus, our inverse systems are very useful since they can simply computes on the rewrite systems even if the functions have some conditions. As another way, for a rule $f(p_1, ..., p_n) \to r \Leftarrow \mathcal{C}ond_0$ in the input deterministic constructor 4-CTRS, we leave $\mathcal{C}ond_0$ in the last of condition part of the conditional rewrite rule obtained by $\mathcal{InvRule}_{\mathsf{N}}(f(p_1, ..., p_n) \to r)$ or $\mathcal{InvRule}_{\mathsf{I}}(f(p_1, ..., p_n) \to r, I)$ where I is an index of f. For example, the case of

$$
\begin{aligned}
R_{11} = R_{12} \cup \\
\{\quad & n_{\emptyset}^{\#}(y_1) \to U_1^1(u_1{}_{\emptyset}^{\#}(y_1), y_1), \quad n_{\emptyset}^{\#}(n(x_d, x_m, x_y)) \to tp_3(x_d, x_m, x_y), \\
& U_1^1(tp_4(y_2, x_d, x_m, x_y), y_1) \to U_2^1(leq_{\{1\}}^{\#}(y_2, 0), x_d, x_m, x_y, y_1, y_2), \\
& U_2^1(x_d, x_d, x_m, x_y, y_1, y_2) \to tp_3(x_d, x_m, x_y), \\
& u_1{}_{\emptyset}^{\#}(y_1) \to U_1^2(u_2{}_{\emptyset}^{\#}(y_1), y_1), \quad u_1{}_{\emptyset}^{\#}(u_1(x_0, x_d, x_m, x_y)) \to tp_3(x_0, x_d, x_m, x_y), \\
& U_1^2(tp_4(y_2, x_d, x_m, x_y), y_1) \to U_2^2(less_{\{2\}}^{\#}(y_2, s^{30}(0)), x_d, x_m, x_y, y_1, y_2), \\
& U_2^2(x_d, x_d, x_m, x_y, y_1, y_2) \to tp_4(\mathtt{T}, x_d, x_m, x_y), \\
& u_2{}_{\emptyset}^{\#}(y_1) \to U_1^3(u_3{}_{\emptyset}^{\#}(y_1), y_1), \quad u_2{}_{\emptyset}^{\#}(u_2(x_0, x_d, x_m, x_y)) \to tp_3(x_0, x_d, x_m, x_y), \\
& U_1^3(tp_4(y_2, x_d, x_m, x_y), y_1) \to U_2^3(leq_{\{1\}}^{\#}(y_2, 0), x_d, x_m, x_y, y_1, y_2), \\
& U_2^3(x_m, x_d, x_m, x_y, y_1, y_2) \to tp_4(\mathtt{T}, x_d, x_m, x_y), \\
& u_3{}_{\emptyset}^{\#}(y_1) \to U_1^4(u_4{}_{\emptyset}^{\#}(y_1), y_1), \quad u_3{}_{\emptyset}^{\#}(u_3(x_0, x_d, x_m, x_y)) \to tp_3(x_0, x_d, x_m, x_y), \\
& U_1^4(tp_4(y_2, x_d, x_m, x_y), y_1) \to U_2^4(less_{\{2\}}^{\#}(y_2, s^{12}(0)), x_d, x_m, x_y, y_1, y_2), \\
& U_2^4(x_m, x_d, x_m, x_y, y_1, y_2) \to tp_4(\mathtt{T}, x_d, x_m, x_y), \\
& u_4{}_{\emptyset}^{\#}(y_1) \to U_1^5(add_{\emptyset}^{\#}(y_1), y_1), \quad u_4{}_{\emptyset}^{\#}(u_4(x_0, x_d, x_m, x_y)) \to tp_3(x_0, x_d, x_m, x_y), \\
& U_1^5(tp_2(y_2, y_3), y_1) \to U_2^5(add_{\emptyset}^{\#}(y_2), y_1, y_2, y_3), \\
& U_2^5(tp_2(x_d, y_4), y_1, y_2, y_3) \to U_3^5(mult_{\{1\}}^{\#}(y_4, s^{30}(0)), x_d, y_1, y_2, y_3, y_4), \\
& U_3^5(x_m, x_d, y_1, y_2, y_3, y_4) \to U_4^5(mult_{\{1\}}^{\#}(y_3, s^{360}(0)), x_d, x_m, y_1, y_2, y_3, y_4), \\
& U_4^5(x_y, x_d, x_m, y_1, y_2, y_3, y_4) \to tp_4(\mathtt{T}, x_d, x_m, x_y), \\
& leq_{\{1\}}^{\#}(\mathtt{T}, 0) \to 0, \ \ leq_{\{1\}}^{\#}(\mathtt{T}, 0) \to s(y), \ \ leq_{\{1\}}^{\#}(\mathtt{F}, s(x)) \to 0, \ \ U_1^6(y, x, z) \to s(y), \\
& leq_{\{1\}}^{\#}(z, s(x)) \to U_1^6(leq_{\{1\}}^{\#}(z, x), x, z), \quad leq_{\{1\}}^{\#}(leq(x, y), x) \to y, \\
& less_{\{2\}}^{\#}(\mathtt{F}, 0) \to 0, \ less_{\{2\}}^{\#}(\mathtt{T}, s(y)) \to 0, \ less_{\{2\}}^{\#}(\mathtt{F}, 0) \to s(x), \ U_1^7(x, y, z) \to s(x), \\
& less_{\{2\}}^{\#}(z, s(y)) \to U_1^7(less_{\{2\}}^{\#}(z, y), y, z), \quad less_{\{2\}}^{\#}(less(x, y), y) \to x \qquad \}.
\end{aligned}
$$

$$
\begin{aligned}
R_{12} = \{\quad & add_{\emptyset}^{\#}(y) \to tp_2(0, y), \quad add_{\emptyset}^{\#}(add(x, y)) \to tp_2(x, y), \\
& add_{\emptyset}^{\#}(s(z)) \to U_1^8(add_{\emptyset}^{\#}(z), z), \quad U_1^8(tp_2(x, y), z) \to tp_2(s(x), y), \\
& mult_{\{1\}}^{\#}(0, 0) \to y, \quad mult_{\{1\}}^{\#}(0, x) \to 0, \\
& mult_{\{1\}}^{\#}(s(z), s(x)) \to U_1^9(add_{\emptyset}^{\#}(z), x, z), \\
& U_1^9(tp_2(w, y), x, z) \to U_2^9(mult_{\{1\}}^{\#}(w, x), x, y, z, w), \\
& U_2^9(s(y), x, y, z, w) \to s(y), \quad mult_{\{1\}}^{\#}(mult(x, y), x) \to y \qquad \}.
\end{aligned}
$$

Figure 7.1: Definition of $\mathcal{I}nv\mathcal{TRS}_{\mathsf{I}}(\mathcal{U}(R_9), \{n_{\emptyset}\})$.

$$
\begin{array}{l}
R_{13} = R_{12} \cup R_{10} \\
\{ \quad n^{\#}_{\emptyset}(y_1) \to U^n_1(add^{\#}_{\emptyset}(y_1), y_1), \qquad n^{\#}_{\emptyset}(n(x_d, x_m, x_y)) \to tp_3(x_d, x_m, x_y), \\
\quad U^n_1(tp_2(y_2, y_3), y_1) \to U^n_2(add^{\#}_{\emptyset}(y_2), y_1, y_2, y_3), \\
\quad U^n_2(tp_2(x_d, y_4), y_1, y_2, y_3) \to U^n_3(mult^{\#}_{\{1\}}(y_4, s^{30}(0)), x_d, y_1, y_2, y_3, y_4), \\
\quad U^n_3(x_m, x_d, y_1, y_2, y_3, y_4) \to U^n_4(mult^{\#}_{\{1\}}(y_3, s^{360}(0)), x_d, x_m, y_1, y_2, y_3, y_4), \\
\quad U^n_4(x_y, x_d, x_m, y_1, y_2, y_3, y_4) \to U^n_5(leq(0, x_d), x_d, x_m, x_y, y_1, y_2, y_3, y_4), \\
\quad U^n_5(\mathtt{T}, x_d, x_m, x_y, y_1, y_2, y_3, y_4) \to U^n_6(less(x_d, s^{30}(0)), x_d, x_m, x_y, y_1, y_2, y_3, y_4), \\
\quad U^n_6(\mathtt{T}, x_d, x_m, x_y, y_1, y_2, y_3, y_4) \to U^n_7(leq(0, x_m), x, y, z, y_1, y_2, y_3, y_4, y_5), \\
\quad U^n_7(\mathtt{T}, x_d, x_m, x_y, y_1, y_2, y_3, y_4) \to U^n_8(less(x_m, s^{12}(0)), x_d, x_m, x_y, y_1, y_2, y_3, y_4), \\
\quad U^n_8(\mathtt{T}, x_d, x_m, x_y, y_1, y_2, y_3, y_4) \to tp_3(x_d, x_m, x_y), \qquad \}.
\end{array}
$$

Figure 7.2: Definition of R_{13}.

n is as follows:

$$
\begin{array}{l}
n^{\#}_{\emptyset}(y_1) \to tp_3(x_d, x_m, x_y) \\
\qquad \Leftarrow add^{\#}_{\emptyset}(y_1) \to tp_2(y_2, y_3) \wedge add^{\#}_{\emptyset}(y_2) \to tp_2(x_d, y_4) \\
\qquad\quad \wedge mult^{\#}_{\emptyset}(y_4) \to tp_2(s^{30}(0), x_m) \wedge mult^{\#}_{\emptyset}(y_3) \to tp_2(s^{360}(0), x_y) \\
\qquad\quad \wedge leq(0, x_d) \to \mathtt{T} \wedge less(x_d, s^{30}(0)) \to \mathtt{T} \\
\qquad\quad \wedge leq(0, x_m) \to \mathtt{T} \wedge less(x_m, s^{12}(0)) \to \mathtt{T}.
\end{array}
$$

Then, we can obtain the system R_{13} by $\mathcal{I}nv\mathcal{CTRS}_{\mathsf{I}}$ in Figure 7.2. R_{13} can also search only for the desirable solutions, and terminates on flexible reduction since $mult^{\#}_{\{1\}}$, $add^{\#}_{\emptyset}$, leq and $less$ terminate by flexible reduction.

Chapter 8

Conclusion

In this paper, we have proposed inverse systems of ground-convergent constructor 3-TRSs, and shown a result on termination of generated natural inverse systems. Future works are extending the proposed methods for constructor TRSs to that for all TRSs and also relaxing the input class whose inverse systems terminate. Study on flexible reduction is strongly expected because we may eliminate the first condition in Theorem 7.

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